

# COHOMOLOGY RINGS OF THE PLACTIC MONOID ALGEBRA VIA A GRÖBNER — SHIRSHOV BASIS

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## Abstract

In this paper we calculate the cohomology ring  $\text{Ext}_{\mathbb{k}\text{Pl}_n}^*(\mathbb{k}, \mathbb{k})$  and the Hochschild cohomology ring of the plactic monoid algebra  $\mathbb{k}\text{Pl}_n$  via the Anick resolution using a Gröbner — Shirshov basis.

## Introduction

The plactic monoid was discovered by Knuth [15], who used an operation given by Schensted in his study of the longest increasing subsequence of a permutation. It was named and systematically studied by Lascoux and Schützenberger [22], who allowed any totally ordered alphabet in the definition. It is known that the elements of plactic monoid can be written in the canonical form, and in this form can be identified with some type of the Young tableaux. Because of its strong relations to Young tableaux, the plactic monoid has already become a classical tool in several areas of representation theory and algebraic combinatorics [20].

Among the significant applications are: a proof of the Littlewood–Richardson rule, an algorithm which allows to decompose tensor product of representations of unitary groups, a combinatorial description of the Kostka — Foulkes polynomials, which arise as entries of character table of the finite linear group. The plactic monoid appeared also in: theory of modular representations of the symmetric group, quantum groups, via the representation theory of quantum enveloping algebras. It is worth mentioning that even though the combinatorics of the plactic monoid has been extensively studied, there are only a few preliminary results of the corresponding plactic algebra over a field [9].

The plactic monoid was connected to the free 2-nilpotent Lie algebra (which is a subalgebra of the parastatistics algebra) in the work [24] by J.-L. Loday and T. Popov. The connection is through the quantum deformation (in the sense of Drinfeld) of the parastatistics algebra. But the first work where the connection between the plactic monoid algebra and parastatistics algebra (in two dimensions) was found is [30].

In [5] there was given an independent proof of uniqueness of (Robinson — Shensted) Knuth normal forms of elements of (Knuth — Schützenberger) plactic monoid.

Gröbner bases and Gröbner — Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [17, 28], free Lie algebras [27, 28] by H. Hironaka [13] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [6] for ideals of the polynomial algebras.

Cain et al [7] use the Schensted–Knuth normal form (the set of (semistandard) Young tableaux) to prove that the multiplication table of column words (strictly decreasing words with respect to some order on the letters) forms a finite Gröbner — Shirshov basis of the finitely generated plactic monoid. In [5] were given new explicit formulas for the multiplication tables of row (nondecreasing word) and column words and independent proofs that the resulting sets of relations are Gröbner — Shirshov bases in row and column generators respectively.

The Anick resolution was obtained by David J. Anick in 1986 [1]. This is a resolution for a field  $\mathbb{k}$  considered as an  $A$ -module, where  $A$  is an associative augmented algebra over  $\mathbb{k}$ . This resolution reflects the combinatorial properties of  $A$  because it is based on the Composition–Diamond Lemma [3, 2]; i.e., Anick defined the set of  $n$ -chains via the leading terms of a Gröbner — Shirshov basis [17, 28, 4] (Anick called it the set of obstructions), and the differentials are defined inductively via  $\mathbb{k}$ -module splitting maps, the leading terms and the normal forms of words.

Later Yuji Kobayashi [14] obtained the resolution for a monoid algebra presented by a complete rewriting system. He constructed an effective free acyclic resolution of modules over the algebra of the monoid whose chains are given by paths in the graph of reductions. These chains are a particular case of chains defined by Anick [1], and differentials have “Anick’s spirit”, i.e., the differentials are described inductively via contracting homotopy, leading terms and normal forms. Further Philippe Malbos [16] constructed a free acyclic resolution in the same spirit for  $R\mathcal{C}$  as a  $\mathcal{C}$ -bimodule over a commutative ring  $R$ , where  $\mathcal{C}$  is a small category endowed with a convergent presentation. The resolution is constructed with the use of the additive Kan extension of the Anick antichains generated by a set of normal forms. This construction can be adapted

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to the construction of the analogous resolution for internal monoids in a monoidal category admitting a finite convergent presentation. Malbos also showed (using the resolution) that if a small category admits a finite convergent presentation then its Hochschild–Mitchell homology is of finite type in all degrees.

The Anick resolution can be extended to the case of operads. The correspondence technique has been developed by Vladimir Dotsenko and Anton Khoroshkin [11].

Michael Jöllenbeck, Volkmar Welker [21] and independently of them Emil Scöldbberg [29] developed a new technique "Algebraic Discrete Morse Theory". In particular, this technique makes it possible to describe the differentials of the Anick resolution; in fact, we have a very useful machinery for constructing homotopy equivalent complexes just using directed graphs. Algebraic Discrete Morse Theory is algebraic version of Forman's Discrete Morse theory [18], [19]. Discrete Morse theory allows to construct, starting from a (regular) CW-complex, a new homotopy-equivalent CW-complex with fewer cells.

In this paper, we use this technique (the Jöllenbeck — Scöldbberg — Welker machinery) for calculating the cohomology ring and the Hochschild cohomology ring of the plactic monoid algebra.

## 1 Preliminaries.

Let us recall some definitions and the basic concept of Algebraic Discrete Morse theory [21], [29].

**Basic concept.** Let  $R$  be a ring and  $C_\bullet = (C_i, \partial_i)_{i \geq 0}$  be a chain complex of free  $R$ -modules  $C_i$ . We choose a basis  $X = \cup_{i \geq 0} X_i$  such that  $C_i \cong \bigoplus_{c \in X_i} Rc$ . Write the differentials  $\partial_i$  with respect to the basis  $X$  in the following form:

$$\partial_i : \begin{cases} C_i \rightarrow C_{i-1} \\ c \mapsto \partial_i(c) = \sum_{c' \in X_{i-1}} [c : c'] \cdot c'. \end{cases}$$

Given a complex  $C_\bullet$  and a basis  $X$ , we construct a directed weighted graph  $\Gamma(C) = (V, E)$ . The set of vertices  $V$  of  $\Gamma(C)$  is the basis  $V = X$  and the set  $E$  of weighted edges is given by the rule

$$(c, c', [c : c']) \in E \quad \text{iff} \quad c \in X_i, c' \in X_{i-1}, \text{ and } [c : c'] \neq 0.$$

**Definition 1.1.** A finite subset  $\mathcal{M} \subset E$  in the set of edges is called an *acyclic matching* if it satisfies the following three conditions:

- (Matching) Each vertex  $v \in V$  lies in at most one edge  $e \in \mathcal{M}$ .
- (Invertibility) For all edges  $(c, c', [c : c']) \in \mathcal{M}$  the weight  $[c : c']$  lies in the center  $Z(R)$  of the ring  $R$  and is a unit in  $R$ .
- (Acyclicity) The graph  $\Gamma_{\mathcal{M}}(V, E_{\mathcal{M}})$  has no directed cycles, where  $E_{\mathcal{M}}$  is given by

$$E_{\mathcal{M}} := (E \setminus \mathcal{M}) \cup \{(c', c, [c : c']^{-1}) \text{ with } (c, c', [c : c']) \in \mathcal{M}\}.$$

For an acyclic matching  $\mathcal{M}$  on the graph  $\Gamma(C_\bullet) = (V, E)$ , we introduce the following notation, which is an adaption of the notation introduced in [18] to our situation.

- We call a vertex  $c \in V$  *critical* with respect to  $\mathcal{M}$  if  $c$  does not lie in an edge  $e \in \mathcal{M}$ ; we write

$$X_i^{\mathcal{M}} := \{c \in X_i : c \text{ critical}\}$$

for the set of all critical vertices of homological degree  $i$ .

- We write  $c' \leq c$  if  $c \in X_i$ ,  $c' \in X_{i-1}$ , and  $[c : c'] \neq 0$ .
- $\text{Path}(c, c')$  is the set of paths from  $c$  to  $c'$  in the graph  $\Gamma_{\mathcal{M}}(C_\bullet)$ .
- The weight  $\omega(p)$  of a path  $p = c_1 \rightarrow \dots \rightarrow c_r \in \text{Path}(c_1, c_r)$  is given by

$$\omega(c_1 \rightarrow \dots \rightarrow c_r) := \prod_{i=1}^{r-1} \omega(c_i \rightarrow c_{i+1}),$$

$$\omega(c \rightarrow c') := \begin{cases} -\frac{1}{[c : c']}, & c \leq c', \\ [c : c'], & c' \leq c. \end{cases}$$

- We write  $\Gamma(c, c') := \sum_{p \in \text{Path}(c, c')} \omega(p)$  for the sum of weights of all paths from  $c$  to  $c'$ .

**Theorem 1.1.** [21, Theorem 2.2] *The chain complex  $(C_\bullet, \partial_\bullet)$  of free  $R$ -modules is homotopy-equivalent to the complex  $(C_\bullet^{\mathcal{M}}, \partial_\bullet^{\mathcal{M}})$  which is complex of free  $R$ -modules and*

$$C_i^{\mathcal{M}} := \bigoplus_{c \in X_i^{\mathcal{M}}} Rc,$$

$$\partial_i^{\mathcal{M}} : \begin{cases} C_i^{\mathcal{M}} \rightarrow C_{i-1}^{\mathcal{M}} \\ c \mapsto \sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma(c, c') c'. \end{cases}$$

In [21, Appendix B, Lemma B.3], there was constructed a contracting homotopy between the Morse complex and the original complex. We use the same denotations.

**Lemma 1.1.** *Let  $(C_\bullet, \partial_\bullet)$  be a complex of free  $R$ -modules,  $\mathcal{M} \subset E$  a matching on the associated graph  $\Gamma(C_\bullet) = (V, E)$ , and  $(C_\bullet^{\mathcal{M}}, \partial_\bullet^{\mathcal{M}})$  the Morse complex. The following maps define a chain homotopy;*

$$\tilde{h}_\bullet : C_\bullet \rightarrow C_\bullet^{\mathcal{M}}$$

$$X_n \ni c \mapsto h(c) = \sum_{c^{\mathcal{M}} \in X_n^{\mathcal{M}}} \Gamma(c, c^{\mathcal{M}}) c^{\mathcal{M}}, \quad (1.1)$$

$$\hat{h}_\bullet : C_\bullet^{\mathcal{M}} \rightarrow C_\bullet$$

$$X_n^{\mathcal{M}} \ni c^{\mathcal{M}} \mapsto h^{\mathcal{M}}(c^{\mathcal{M}}) = \sum_{c \in X_n} \Gamma(c^{\mathcal{M}}, c) c \quad (1.2)$$

**Morse matching and the Anick resolution.** Throughout this paper,  $\mathbb{k}$  denotes any field and  $\Lambda$  is an associative  $\mathbb{k}$ -algebra with unity and augmentation; i.e., a  $\mathbb{k}$ -algebra homomorphism  $\varepsilon : \Lambda \rightarrow \mathbb{k}$ . Let  $X$  be a set of generators for  $\Lambda$ . Suppose that  $\leq$  is a well ordering on  $X^*$ , the free monoid generated by  $X$ . For instance, given a fixed well ordering on the letters, one may order words “length-lexicographically” by first ordering by length and then comparing words of the same length by checking which of them occurs earlier in the dictionary. Denote by  $\mathbb{k}\langle X \rangle$  the free associative  $\mathbb{k}$ -algebra with unity on  $X$ . There is a canonical surjection  $f : \mathbb{k}\langle X \rangle \rightarrow \Lambda$  once  $X$  is chosen, in other words, we get  $\Lambda \cong \mathbb{k}\langle X \rangle / \ker(f)$

Let  $\text{GSB}_\Lambda$  be a Gröbner–Shirshov basis for  $\Lambda$ . Denote by  $\mathfrak{V}$  the set of the leading terms in  $\text{GSB}_\Lambda$  and let  $\mathfrak{B} = \text{Irr}(\ker(f))$  be the set of irreducible words (not containing the leading monomials of relations as subwords) or  $\mathbb{k}$ -basis for  $\Lambda$  (see CD-Lemma [3, 2]). Following Anick [1], call  $\mathfrak{V}$  the set of obstructions (antichains) for  $\mathfrak{B}$ . For  $n \geq 1$ ,  $v = x_{i_1} \cdots x_{i_t} \in X^*$  is an  $n$ -prechain whenever there exist  $a_j, b_j \in \mathbb{Z}$ ,  $1 \leq j \leq n$ , satisfying

1.  $1 = a_1 < a_2 \leq b_1 < a_3 \leq b_2 < \dots < a_n \leq b_{n-1} < b_n = t$  and,
2.  $x_{i_{a_j}} \cdots x_{i_{b_j}} \in \mathfrak{V}$  for  $1 \leq j \leq n$ .

An  $n$ -prechain  $x_{i_1} \cdots x_{i_t}$  is an  $n$ -chain iff the integers  $\{a_j, b_j\}$  can be chosen so that

3.  $x_{i_1} \cdots x_{i_s}$  is not an  $m$ -prechain for any  $s < b_m$ ,  $1 \leq m \leq n$ .

As in [1], we say that the elements of  $X$  are 0-chains, the elements of  $\mathfrak{V}$  are 1-chains, and denote the set of  $n$ -chains by  $\mathfrak{V}^{(n)}$ .

As usual, the cokernel of a  $\mathbb{k}$ -module map  $\eta : \mathbb{k} \rightarrow \Lambda$  will be denoted as  $\Lambda/\mathbb{k}$ . For each left  $\Lambda$ -module  $C$ , construct the relatively free  $\Lambda$ -module

$$B_n(\Lambda, C) := \Lambda \otimes_{\mathbb{k}} \underbrace{(\Lambda/\mathbb{k}) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} (\Lambda/\mathbb{k})}_{n \text{ factors } \Lambda/\mathbb{k}} \otimes_{\mathbb{k}} C,$$

Define right  $\Lambda$ -module homomorphisms  $\partial_n : B_n \rightarrow B_{n-1}$  for  $n > 0$  by

$$\partial_n([\lambda_1] \dots [\lambda_n]) = \lambda_1[\lambda_2] \dots [\lambda_n] + (-1)^n [\lambda_1] \dots [\lambda_{n-1}] \lambda_n + \sum_{i=1}^{n-1} (-1)^i [\lambda_1] \dots [\lambda_i \lambda_{i+1}] \dots [\lambda_n].$$

As is well known, the chain complex  $(B_\bullet(\Lambda, C), \partial_\bullet)$  is a normalized bar resolution for the left  $\Lambda$ -module  $C$ . We assume that  $C = \mathbb{k}$ , i.e., for the  $c \in \mathbb{k}$ ,  $\lambda \in \Lambda$  we have  $\lambda \cdot c = \varepsilon(\lambda)c$ . Let us rewrite the resolution  $(B_\bullet(\Lambda, \mathbb{k}), \partial_\bullet)$  as

$$B_0 = \Lambda, \quad B_n = \bigoplus_{\omega_1, \dots, \omega_n \in \mathfrak{B}_\Lambda} \Lambda[\omega_1 | \dots | \omega_n], \quad n \geq 1$$

with differentials

$$\partial_n([\omega_1] \dots [\omega_n]) = \varepsilon(\omega_1)[\omega_2] \dots [\omega_n] + (-1)^n [\omega_1] \dots [\omega_{n-1}] \omega_n + \sum_{i=1}^{n-1} (-1)^{n-i} [\omega_1] \dots [f(\omega_i \omega_{i+1})] \dots [\omega_n]. \quad (1.3)$$

**Theorem 1.2** (Jöllenbeck — Scöldbberg — Welker). For  $\omega \in X^*$ , let  $\mathfrak{V}_{\omega,i}$  be the vertices  $[\omega_1 | \dots | \omega_n]$  in  $\Gamma_{B_\bullet(\Lambda, \mathbb{k})}$  such that  $\omega = \omega_1 \dots \omega_n$  and  $i$  is the larger integer  $i \geq -1$  such that  $\omega_1 \dots \omega_{i+1} \in \mathfrak{V}^i$  is an Anick  $i$ -chain. Let  $\mathfrak{V}_\omega = \bigcup_{i \geq -1} \mathfrak{V}_{\omega,i}$ .

Define a partial matching  $\mathcal{M}_\omega$  on  $(\Gamma_{B_\bullet(\Lambda, \mathbb{k})})_\omega = \Gamma_{B_\bullet(\Lambda, \mathbb{k})}|_{\mathfrak{V}_\omega}$  by letting  $\mathcal{M}_\omega$  consist of all edges

$$[\omega_1 | \dots | \omega'_{i+2} | \omega''_{i+2} | \dots | \omega_n] \rightarrow [\omega_1 | \dots | \omega_{i+2} | \dots | \omega_n]$$

when  $[\omega_1 | \dots | \omega_n] \in \mathfrak{V}_{\omega,i}$ , such that  $\omega'_{i+2} \omega''_{i+2} = \omega_{i+2}$  and  $[\omega_1 | \dots | \omega_{i+1} | \omega'_{i+2}] \in \mathfrak{V}^{i+1}$  is an Anick  $(i+1)$ -chain.

The set of edges  $\mathcal{M} = \bigcup_{\omega} \mathcal{M}_\omega$  is a Morse matching on  $\Gamma_{B_\bullet(\Lambda, \mathbb{k})}$ , with critical cells  $X_n^{\mathcal{M}} = \mathfrak{V}^{n-1}$  for all  $n$ .

From this theorem we get the following proposition ([21, Theorem 4.4] and [29, Theorem 4]). But here we assume that  $\varepsilon : \Lambda \rightarrow \mathbb{k}$  is arbitrary.

**Proposition 1.1.** The chain complex  $(A_\bullet(\Lambda), d_\bullet)$  defined by

$$A_n(\Lambda) = \bigoplus_{v \in \mathfrak{V}^{(n-1)}} \Lambda v, \quad d_n(v) = \sum_{c' \in \mathfrak{V}^{(n-2)}} \Gamma(v, v') v'$$

where all paths from graph  $\Gamma_{B_\bullet(\Lambda, \mathbb{k})}^{\mathcal{M}}$ , is the  $\Lambda$ -free Anick resolution for  $\mathbb{k}$ .

Let us demonstrate how the Morse matching machinery work.

**Example 1.1.** Let us consider the following algebra  $\Lambda = \mathbb{k}\langle x, y \rangle / (x^2 - y^2)$ . We set  $x > y$ , then we have

$$xxx \xrightarrow{(-=)} x(xx) \rightarrow xy^2$$

and

$$xxx \xrightarrow{((-)=)} (xx)x \rightarrow y^2x$$

i.e., we have to add the relation  $xy^2 = y^2x$ , using Buchberger — Shirshov's algorithm we get

$$xxy^2 \xrightarrow{(-=)} x(xy^2) \rightarrow x(y^2x) \rightarrow (xy^2)x \rightarrow (y^2x)x \rightarrow y^2xx \rightarrow y^2y^2 \rightarrow y^4$$

in other hand

$$xxy^2 \xrightarrow{((-)=)} (xx)y^2 \rightarrow y^2y^2 \rightarrow y^4$$

Thus we get  $\text{GSB}_\Lambda = \{x^2 - y^2, xxy^2 - y^2x\}$  then we have,

$$\mathfrak{V} = \{x^2, xxy^2\}, \quad \mathfrak{V}^{(2)} = \{x \overline{xx}, x \overline{xy^2}\}, \quad \mathfrak{V}^{(3)} = \{x \overline{xxx}, x \overline{xy^2x}\}, \dots,$$

i.e.,  $\mathfrak{V}^{(\ell)} \mathbb{k} = \text{Span}_{\mathbb{k}}(x^{\ell+1}, x^\ell y^2)$ ,  $\ell \geq 0$ . We will use the bar notations, i.e., we will denote the elements of the set  $\mathfrak{V}^{(\ell)}$  as  $\underbrace{[x | \dots | x]}_{\ell+1}$  and  $\underbrace{[x | \dots | x | y^2]}_{\ell}$ ,  $\ell \geq 0$ . Thus we have the following (exact) chain complex

$$\dots \rightarrow \Lambda \otimes_{\mathbb{k}} \mathfrak{V}^{(\ell)} \mathbb{k} \xrightarrow{d_\ell} \Lambda \otimes_{\mathbb{k}} \mathfrak{V}^{(\ell-1)} \mathbb{k} \xrightarrow{d_{\ell-1}} \dots \xrightarrow{d_2} \Lambda \otimes_{\mathbb{k}} \mathfrak{V} \mathbb{k} \xrightarrow{d_1} \Lambda \otimes_{\mathbb{k}} \text{Span}_{\mathbb{k}}(x, y) \xrightarrow{d_0} \Lambda \xrightarrow{\varepsilon} \mathbb{k} \rightarrow 0.$$

Let us define all differentials via Morse matching machinery. We have to consider the following directed weighted graphs (see fig.1, fig.2 and fig.3).

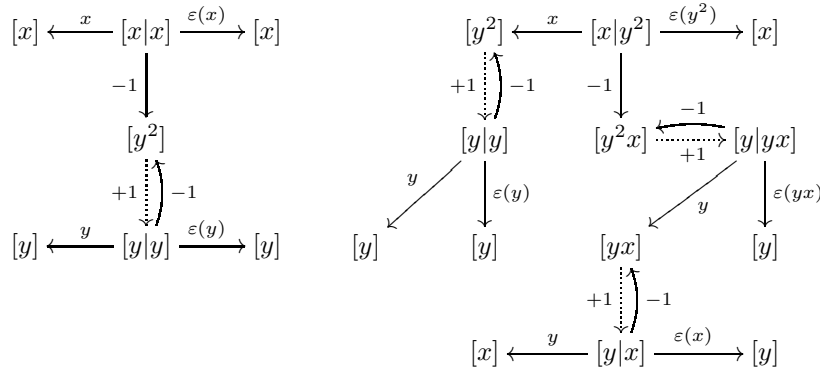


Figure 1: Here is shown the Morse matching, the correspondence edges are shown as dots arrows.

Thus we get

$$d_1[x|x] = x[x] + \varepsilon(x)[x] - y[y] - \varepsilon(y)[y],$$

$$\begin{aligned}
d_1[x|y^2] &= \varepsilon(y^2)[x] + xy[y] + \varepsilon(y)x[y] - \varepsilon(yx)[y] - y^2[x] - \varepsilon(x)y[y], \\
d_\ell[\underbrace{x|\dots|x}_{\ell-1}|y^2] &= x[\underbrace{x|\dots|x}_{\ell-1}|y^2] + (-1)^{\ell+1}\varepsilon(y^2)[\underbrace{x|\dots|x}_{\ell}] + (-1)^\ell y[\underbrace{x|\dots|x}_{\ell}] \\
d_\ell[\underbrace{x|\dots|x}_{\ell+1}] &= x[\underbrace{x|\dots|x}_{\ell}] + (-1)^{\ell+1}\varepsilon(x)[\underbrace{x|\dots|x}_{\ell}] + (-1)^\ell y^2[\underbrace{x|\dots|x}_{\ell-1}].
\end{aligned}$$

here  $\ell > 1$ .

**Remark 1.1.** The same algebra  $\Lambda = \mathbb{k}\langle x, y \rangle / (x^2 - y^2)$  with nice Gröbner — Shirshov bases was considered in [10]. There is a small caveat; David J. Anick has developed his technique for right modules. We consider Anick's resolution for left modules.

**Hochschild (co)homology via the Anick resolution.** Keeping the notation from the previous paragraph. As usual we denote by  $\Lambda^e := \Lambda \otimes_{\mathbb{k}} \Lambda^{\text{op}}$  the enveloping algebra for algebra  $\Lambda$ . Follow [21], [29] we shall see how to construct a free  $\Lambda^e$ -resolution for  $\Lambda$  as a (left) right module.

Here we consider the two-sided bar resolution  $B_\bullet(\Lambda, \Lambda)$  which is an  $\Lambda^e$ -free resolution of  $\Lambda$  where

$$B_n(\Lambda, \Lambda) := \Lambda \otimes_{\mathbb{k}} (\Lambda/\mathbb{k})^{\otimes n} \otimes_{\mathbb{k}} \Lambda \cong \Lambda^e \otimes_{\mathbb{k}} (\Lambda/\mathbb{k})^{\otimes n}.$$

The differential is defined as before:

$$\partial_n([\lambda_1 | \dots | \lambda_n]) = (\lambda_1 \otimes 1)[\lambda_2 | \dots | \lambda_n] + \sum_{i=1}^{n-1} (-1)^i [\lambda_1 | \dots | \lambda_i \lambda_{i+1} | \dots | \lambda_n] + (-1)^n (1 \otimes \lambda_n)[\lambda_1 | \dots | \lambda_{n-1}]. \quad (1.4)$$

Let us consider the same matching  $\mathcal{M} = \bigcup_{\omega} \mathcal{M}_{\omega}$  as before, and get

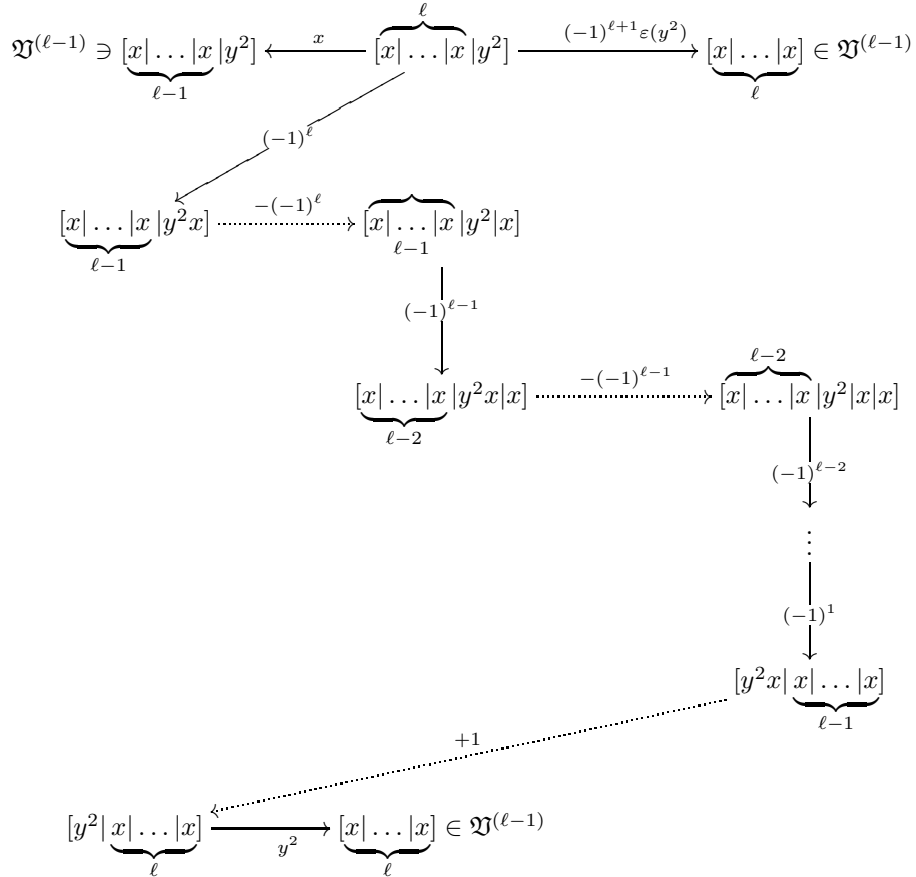


Figure 2: As before the dots arrows mean the edges from Morse matching.

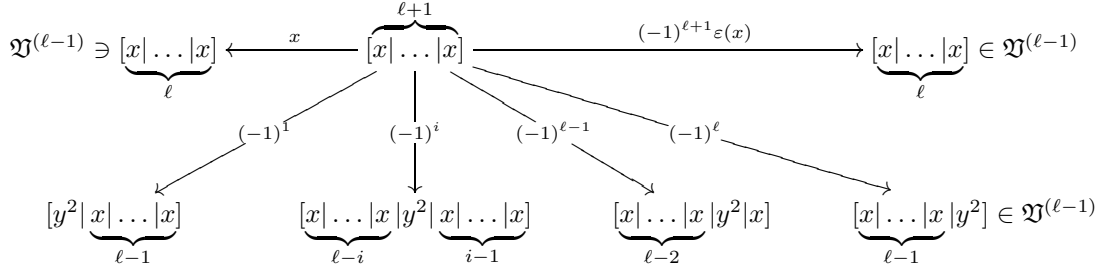


Figure 3: From the previous figure follows that there is only one  $(\ell - 1)$ th Anick's chain in bottom level.

**Proposition 1.2.** [21, Chapter 5], [29, Lemma 9 and Theorem 5] *The set of edges  $\mathcal{M} = \bigcup \mathcal{M}_\omega$  is a Morse matching on  $\Gamma_{B_\bullet(\Lambda, \Lambda)}$ , with Anick chains as critical points. Moreover, the complex  $(HA_\bullet(\Lambda), A\partial_\bullet)$  which is defined as follows:*

$$HA_{n+1}(\Lambda) = \Lambda^e \otimes \mathfrak{V}^{(n)} \mathbb{k}, \quad A\partial_{n+1}(v) = \sum_{v' \in \mathfrak{V}^{(n)}} \Gamma(v, v') v'.$$

is a free  $\Lambda^e$  resolution of  $\Lambda$ .

The  $\Lambda^e$ -resolution defined above will also be denoted by  $A_\bullet(\Lambda)$ . It will always be clear from the context what kind of resolution is being considered.

**Multiplication in Cohomology via Gröbner — Shirshov basis.** Let us consider the cohomological multiplication of associative algebra  $\Lambda$  via Gröbner — Shirshov basis  $\text{GSB}_\Lambda$ . From [8, §7, Chapter IX] we know that first of all we need a map

$$g_\bullet : B_\bullet(\Lambda \otimes \Lambda) \rightarrow B_\bullet(\Lambda) \otimes B_\bullet(\Lambda)$$

which is given by the formula

$$g_n[\lambda_1 \otimes \lambda'_1 | \dots | \lambda_n \otimes \lambda'_n] = \sum_{0 \leq p \leq n} [\lambda_1 | \dots | \lambda_p] \lambda_{p+1} \dots \lambda_n \otimes \lambda'_1 \dots \lambda'_p [\lambda'_{p+1} | \dots | \lambda'_n]. \quad (1.5)$$

Let us rewrite this formulae in the following way:

$$g_n[\lambda \otimes \lambda']_n = \sum_{0 \leq p \leq n} [\lambda]_{1,p} (\lambda')_{p+1,n} \otimes (\lambda)_{1,p} [\lambda']_{p+1,n}, \quad (1.6)$$

here  $[\lambda \otimes \lambda']_n = [\lambda_1 \otimes \lambda'_1 | \dots | \lambda_n \otimes \lambda'_n]$ ,  $[\lambda]_{i,j} := [\lambda_i | \dots | \lambda_j]$ ,  $(\lambda)_{i,j} = \lambda_i \dots \lambda_j$ , for  $i \leq j$  and we put that  $[\lambda]_{i,j} = [\lambda]_{n+1} = \emptyset$ ,  $(\lambda)_{i,j} = ()$  if  $i > j$ . Let us consider the following diagram

$$\begin{array}{ccc} B_\bullet(\Lambda) & \xrightarrow{g_\bullet} & B_\bullet(\Lambda) \otimes B_\bullet(\Lambda) \\ \uparrow \hat{h}_\bullet & & \downarrow \hat{h}_\bullet \otimes \hat{h}_\bullet \\ A_\bullet(\Lambda) & \xrightarrow{Ag_\bullet} & A_\bullet(\Lambda) \otimes A_\bullet(\Lambda) \end{array}$$

from Lemma 1.1 follows that this diagram is commutative. Then using (1.6), (1.1), (1.2) we get

$$\begin{aligned} Ag_n[\nu \otimes \nu']_n &= \\ &= \sum_{\substack{0 \leq p \leq n \\ [\lambda \otimes \lambda']_n \in (\Lambda \otimes \Lambda)^{\otimes n+1} \\ [v]_p \in \mathfrak{V}^{(p-1)}, \\ [u]_{n-p} \in \mathfrak{V}^{n-p-1}}} \Gamma([\nu \otimes \nu'], [\lambda \otimes \lambda']) \Gamma([\lambda]_{1,p}, [v]_p) [v]_p (\lambda')_{p+1,n} \otimes (\lambda)_{1,p} \Gamma([\lambda']_{p+1,n}, [u]_{n-p}) [u]_{n-p} \end{aligned} \quad (1.7)$$

Suppose now we have a Hopf algebra  $H = (H, \Delta_H, \nabla_H, \varepsilon, \eta)$  with comultiplication  $\Delta_H(x) = x \otimes x$  and assume we know Gröbner — Shirshov basis  $\text{GSB}_H$  for algebra  $(H, \nabla_H, \eta)$ , then for some left  $H$ -module  $M$ , we get a following commutative diagram

$$\begin{array}{ccc} \text{Hom}_H(A_p(H), M) \otimes \text{Hom}_H(A_q(H), M) & \xrightarrow{\vee} & \text{Hom}_{H \otimes H} \left( \bigoplus_{r+s=p+q} A_r(H) \otimes A_s(H), M \otimes M \right) \\ & \searrow & \downarrow \Delta_H^* \\ & & \text{Hom}_H(A_{p+q}(H), M) \end{array}$$

where  $\vee$ -product [8, §7, Chapter IX] is given by the following formulae,

$$(\vartheta \vee \vartheta')(c \otimes c') := \vartheta(c) \otimes \vartheta'(c'),$$

then using (1.7) we can describe  $\smile$ -multiplication by the following formulae

$$(\vartheta_p \smile \vartheta_q)([v]_{p+q}) = \sum_{\substack{[\lambda \otimes \lambda']_{n \in (\Lambda \otimes \Lambda)}^{\otimes n+1} \\ [v]_p \in \mathfrak{Y}^{(p-1)}, \\ [u]_q \in \mathfrak{Y}^{q-1}}} \Gamma([\nu \otimes \nu'], [\lambda \otimes \lambda']) \Gamma([\lambda]_{1,p}, [v]_p) \vartheta_p([v]_p) (\lambda')_{p+1,p+q} (\lambda)_{1,p} \Gamma([\lambda']_{p+1,q}, [u]_q) \vartheta_q([u]_q) \quad (1.8)$$

**Remark 1.2.** Since the comultiplication  $\Delta(x) = x \otimes x$  is cocommutative, then  $\vartheta_p \smile \vartheta_q = (-1)^{pq} \vartheta_q \smile \vartheta_p$ , it can allow to simplify the (1.8).

## 2 The Plactic Monoid with Column Generators

In this section, we present an elegant algorithm proposed by C. Schensted. We will also see that there is some connection between Schensted's column algorithm and braids.

**Definition 2.1.** Let  $A = \{1, 2, \dots, n\}$  with  $1 < 2 < \dots < n$ . Then we call  $\text{Pl}(A) := A^* / \equiv$  the plactic monoid on the alphabet set  $A$ , where  $A^*$  is the free monoid generated by  $A$ ,  $\equiv$  is the congruence of  $A^*$  generated by Knuth relations  $\Omega$  and  $\Omega$  consists of

$$ikj = kij \ (i \leq j < k), \quad jki = jik \ (i < j \leq k).$$

For a field  $\mathbb{k}$ , denote by  $\mathbb{k}\text{Pl}(A)$  or by  $\mathbb{k}\text{Pl}_n$  the plactic monoid algebra over  $\mathbb{k}$  of  $\text{Pl}(A)$ .

**Definition 2.2.** A strictly decreasing word  $w \in A^*$  is called a column [7]. Denote the set of columns by  $I$ . Let  $a \in I$  be a column and  $a_i$  the number of the letter  $i$  in  $a$ . Then  $a_i \in \{0, 1\}$ ,  $i = \{1, 2, \dots, n\}$ . Put  $a = (a_1; \dots; a_n)$ . Also we will consider any column as an ordered set  $\{a\} := \{a_{i_1}, \dots, a_{i_\ell}\}$ , here  $\{a_{i_j}\} = \emptyset$  iff  $a_{i_j} = 0$  and we will denote it by  $a = e_{i_1, \dots, i_\ell}$ . Denote by  $e_\emptyset$  the empty column (the unity of the plactic monoid). Also by  $e_i$  we denote the column  $(0; \dots; 0; 1; 0; \dots; 0)$  where 1 is in  $i$ th place.

For example, the word  $a = 875421$  is a column, and we have  $a = (1; 1; 0; 1; 1; 0; 1; 1; 0; \dots; 0)$ ,  $\{a\} = \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}\}$ .

**Definition 2.3** (Schensted's column algorithm). Let  $a \in I$  be a column and let  $x \in A$ .

$$x \cdot a = \begin{cases} xa, & \text{if } xa \text{ is a column;} \\ a' \cdot y, & \text{otherwise} \end{cases}$$

where  $y$  is the rightmost letter in  $a$  and is larger than or equal to  $x$ , and  $a'$  is obtained from  $a$  by replacing  $y$  with  $x$ . We say that an element  $y$  is connected to  $x$  or simply that elements  $y, x$  are connected. And we will use the notation

$$x \rightleftharpoons y := \begin{cases} 1, & \text{iff } x \text{ is connected to } y, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2.4.** Consider two columns  $a, b \in I$  as ordered sets  $\{a\}, \{b\}$  and consider the columns

$$\{b^a\} := \{x \in \{b\} : (y \rightleftharpoons x) = 0 \text{ for any } y \in \{a\}\},$$

$$\{b_a\} := \{x \in \{b\} : (y \rightleftharpoons x) = 1 \text{ for some } y \in \{a\}\}.$$

Introduce binary operations  $\vee, \wedge : I \times I \rightarrow I$  by the formulas:

$$\{a \vee b\} := \{a\} \cup \{b^a\}, \quad \{a \wedge b\} := \{b_a\},$$

then from Schensted's column algorithm it follows that  $a \cdot b = (a \vee b) \cdot (a \wedge b)$ .

**Example 2.1.** Consider the following columns as ordered sets (see fig.4):  $\{a\} = \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$  and  $\{b\} = \{b_{j_1}, b_{j_2}, b_{j_3}, b_{j_4}, b_{j_5}, b_{j_6}\}$ , we also have  $a = e_{i_1, i_2, i_3, i_4}$  and  $b = e_{j_1, j_2, j_3, j_4, j_5, j_6}$ . We get

$$(a_{i_1} \rightleftharpoons b_{j_2}) = 1, \quad (a_{i_2} \rightleftharpoons b_{j_3}) = 1, \quad (a_{i_3} \rightleftharpoons b_{j_6}) = 1,$$

then

$$\{a \vee b\} = \{b_{j_1}, a_{i_1}, a_{i_2}, b_{j_4}, b_{j_5}, a_{i_3}, a_{i_4}\}, \quad \{a \wedge b\} = \{b_{j_2}, b_{j_3}, b_{j_6}\}.$$

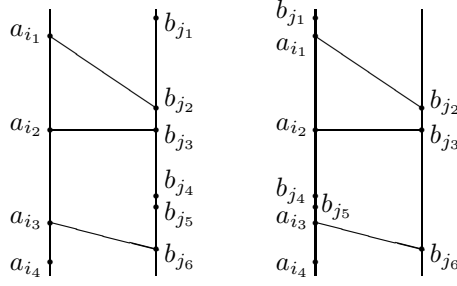


Figure 4: Here is shown  $a \cdot b = (a \vee b) \cdot (a \wedge b)$ .

**Theorem 2.1.** *The triple  $(I, \vee, \wedge)$  with binary operations  $\vee$  and  $\wedge$  satisfies the following equations for any columns  $a, b, c \in I$ :*

$$a \vee b = a \vee c \text{ and } a \wedge b = a \wedge c \text{ imply } b = c, \quad (2.9)$$

$$a \vee d = b \vee d \text{ and } a \wedge d = b \wedge d \text{ imply } a = b, \quad (2.10)$$

$$a \vee b = a \text{ iff } a \wedge b = b \text{ and } a \wedge b = a \text{ iff } a \vee b = b, \quad (2.11)$$

$$a \vee a = a, \quad a \wedge a = a, \quad (2.12)$$

$$a \vee (a \wedge b) = a = a \wedge (a \vee b), \quad (a \wedge b) \vee b = b = (a \vee b) \wedge b, \quad (2.13)$$

$$(a \vee b) \vee ((a \wedge b) \vee c) = a \vee (b \vee c), \quad (2.14)$$

$$(a \vee b) \wedge ((a \wedge b) \vee c) = (a \wedge (b \vee c)) \vee (b \wedge c), \quad (2.15)$$

$$(a \wedge (b \vee c)) \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad (2.16)$$

$$a \vee b = b \iff a \wedge b = a. \quad (2.17)$$

*Proof.* 1.  $a \vee b = a \vee c$  and  $a \wedge b = a \wedge c$  imply  $b = c$ .

From  $a \vee b = a \vee c$  it follows that  $\{b^a\} = \{c^a\}$ . On the other hand, from  $a \wedge b = a \wedge c$  we obtain  $\{b_a\} = \{c_a\}$ , i.e.  $b = c$ .

2.  $a \vee d = b \vee d$  and  $a \wedge d = b \wedge d$  imply  $a = b$ .

Suppose that  $\{x \vee y\} = L_x^f \cup L_x^c \cup R_y^f$ . Here  $L_x^f := \{\chi \in \{x\} : (\chi \rightleftharpoons u) = 0 \text{ for any } u \in \{y\}\}$ ,  $L_x^c := \{\chi \in \{x\} : (\chi \rightleftharpoons u) = 1 \text{ for some } u \in \{u\}\}$  and  $R_y^f := \{u \in \{u\} : (\chi \rightleftharpoons u) = 0 \text{ for all } \chi \in \{x\}\}$ . Consider also the set  $R_y^c := \{u \in \{u\} : (\chi \rightleftharpoons u) = 1 \text{ for some } \chi \in \{x\}\}$ . We have  $\{x\} = L_x^f \cup L_x^c$  and  $\{y\} = R_y^f \cup R_y^c$ .

From  $a \vee d = b \vee d$  we get  $L_a^f \cup L_a^c \cup R_d^f = L_b^f \cup L_b^c \cup R_d^f$ . Further, from  $a \wedge c = b \wedge c$  it follows that  $2R_d^c = R_d^c$  but from  $\{d\} = R_d^f \cup R_d^c = R_d^f \cup R_d^c$  we get  $R_d^f = R_d^c$ , then  $L_a^f \cup L_a^c \cup R_d^f = L_b^f \cup L_b^c \cup R_d^f$  implies that  $L_a^f \cup L_a^c = L_b^f \cup L_b^c$ , i.e.  $a = b$ .

3.  $a \vee b = a$  iff  $a \wedge b = b$  and  $a \wedge b = a$  iff  $a \vee b = b$ .

From  $a \vee b = a$  we conclude that  $\{b^a\} = \emptyset$ , and so  $\{b\} = \{b_a\}$  and vice versa. If  $a \wedge b = a$  then  $\{b_a\} = \{a\}$ , and we infer that  $\{a \vee b\} = \{a\} \cup \{b^a\} = \{b_a\} \cup \{b^a\} = \{b\}$  and vice versa.

5.  $a \vee (a \wedge b) = a = a \wedge (a \vee b)$ ,  $(a \wedge b) \vee b = b = (a \vee b) \wedge b$

From  $\{a \wedge b\} := \{b_a\}$  it follows that  $\{(a \wedge b)^a\} = \emptyset$ ; i.e.  $\{a \vee (a \wedge b)\} = \{a\}$ . Further,  $\{a \vee b\} := \{a\} \cup \{b^a\}$  yields  $\{(a \wedge b)_a\} = \{a\}$ .

Observe that  $\{b_{a \wedge b}\} = \{b_a\}$  and  $\{b^{a \wedge b}\} = \{b^a\}$ ; i.e.,  $\{(a \wedge b) \vee b\} = \{b\}$ . Note that  $\{b^{a \vee b}\} = \{b^a\} \cap \{b^{b^a}\}$  and let us prove that  $\{b^{a \vee b}\} = \emptyset$ . Now,  $\{b^{b^a}\} = \{b_a\}$ , from  $\{b^a\} \cap \{b_a\} = \emptyset$  we get  $\{b^{a \vee b}\} = \emptyset$ , and so  $\{b_{a \vee b}\} = \{b\}$ , i.e.  $\{(a \vee b) \wedge b\} = \{b\}$ .

6.  $(a \vee b) \wedge ((a \wedge b) \vee c) = (a \wedge (b \vee c)) \vee (b \wedge c)$ .

First of all we need to describe the column  $(a \wedge (b \vee c))$ . Suppose that  $x_\alpha \in \{a\}$ ,  $y_\beta \in \{b\}$  and  $z_\gamma \in \{c\}$  are such that  $(x_\alpha \rightleftharpoons y_\beta) = 1$ ,  $(x_\alpha \rightleftharpoons z_\gamma) = 1$  and  $(y_\beta \rightleftharpoons z_\gamma) = 0$ ; then  $\min\{y_\beta, z_\gamma\} \in \{a \wedge (b \vee c)\}$ .

Let  $x_\gamma \in \{(a \vee b) \wedge ((a \wedge b) \vee c)\}$ . Then  $x_\gamma \in \{(a \wedge b) \vee c\}$  and there exists  $y_\beta \in \{a \vee b\}$  such that  $(y_\beta \rightleftharpoons x_\gamma) = 1$ . If  $x_\gamma \in \{a \wedge b\}$  then there's no  $z_\rho \in \{c^{a \wedge b}\}$  such that  $\beta \leq \gamma \leq \rho$  then  $x_\gamma \in \{a \wedge (b \vee c)\}$ , i.e.,  $x_\gamma \in \{(a \wedge (b \vee c)) \vee (b \wedge c)\}$ . Let  $x_\gamma \in \{c^{(a \wedge b)}\}$  then there is no  $x_\rho \in \{a \wedge b\}$  such that  $\beta \leq \rho \leq \gamma$ , and



hence  $x_\gamma \in \{a \wedge (b \vee c)\}$ , i.e.,  $x_\gamma \in \{(a \wedge (b \vee c)) \vee (b \wedge c)\}$ . We have proved that  $\{(a \vee b) \wedge ((a \wedge b) \vee c)\} \subseteq \{(a \wedge (b \vee c)) \vee (b \wedge c)\}$ .

Let  $x_\gamma \in \{(a \wedge (b \vee c)) \vee (b \wedge c)\}$ . If  $x_\gamma \in \{(a \wedge (b \vee c))\}$  then  $x_\gamma \in \{b \vee c\}$  and there exists  $y_\alpha \in \{a\}$  with  $(y_\alpha \rightleftharpoons x_\gamma) = 1$ . Assume that  $x_\gamma \in \{b\}$ . Then there is no  $z_\beta \in \{c\}$  such that  $\alpha \leq \gamma \leq \beta$ . Therefore,  $x_\gamma \in \{(a \wedge b) \vee c\}$ . Since  $y_\alpha \in \{a\}$ , it follows that  $y_\alpha \in \{a \vee b\}$  and, since  $(y_\alpha \rightleftharpoons x_\gamma) = 1$ , we see that  $x_\gamma \in \{(a \vee b) \wedge ((a \wedge b) \vee c)\}$ . Suppose now that  $x_\gamma \in \{c^b\}$ . Then there is no  $z_\rho \in \{b\}$  with  $\alpha \leq \rho \leq \gamma$ , and hence  $x_\gamma \in \{(a \wedge b) \vee c\}$ , and, since there is  $y_\alpha \in \{a\}$ , we get  $x_\gamma \in \{(a \vee b) \wedge ((a \wedge b) \vee c)\}$ . Now, consider the case  $x_\gamma \in \{(b \wedge c)\}$ . There exists  $z_\beta \in \{b\}$  such that  $(z_\beta \rightleftharpoons x_\gamma) = 1$ , and there is no  $y_\alpha \in \{a \wedge (b \vee c)\}$  such that  $(y_\alpha \rightleftharpoons x_\gamma) = 1$ , i.e., if for some  $u_\rho \in \{a\}$  there exists  $x_{\gamma'} \in \{c\}$  with  $(u_\rho \rightleftharpoons x_{\gamma'}) = 1$  then  $\alpha \leq \gamma' < \beta$ ; i.e.,  $z_\beta \in \{b^a\}$ , and hence  $x_\gamma \in \{(a \vee b) \wedge ((a \wedge b) \vee c)\}$ . We have proved that  $\{(a \vee b) \wedge ((a \wedge b) \vee c)\} \supseteq \{(a \wedge (b \vee c)) \vee (b \wedge c)\}$ .

$$7. (a \wedge (b \vee c)) \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

Take  $c_{\gamma_1} \in \{c\}$  and  $c_{\gamma_1} \in \{(a \wedge (b \vee c)) \wedge (b \wedge c)\}$ . Then there exists  $b_{\beta_1} \in \{b\}$  with  $(b_{\beta_1} \rightleftharpoons c_{\gamma_1}) = 1$  and also there exists  $x_{c_{\gamma_1}} \in \{a \wedge (b \vee c)\}$  such that  $(x_{c_{\gamma_1}} \rightleftharpoons c_{\gamma_1}) = 1$  and  $(x_{c_{\gamma_1}} \rightleftharpoons b_{\beta_1}) = 1$ . Since  $x_{c_{\gamma_1}} \in \{a \wedge (b \vee c)\}$ , there exists  $y_\alpha \in \{a\}$  with  $(y_\alpha \rightleftharpoons x_{c_{\gamma_1}}) = 1$ .

Observe that if  $x_{c_{\gamma_1}} \in \{b\}$  then  $(x_{c_{\gamma_1}} \rightleftharpoons c_{\gamma_1}) = 1$  and  $(b_{\beta_1} \rightleftharpoons c_{\gamma_1}) = 1$  but it is possible iff  $x_{c_{\gamma_1}} = b_{\beta_1}$ . This means that  $c_{\gamma_1} \in \{c\}$  is connected with some  $x_{c_{\gamma_1}} \in \{b\}$  which is connected with some  $y_\alpha \in \{a\}$ ; i.e.,  $\{(a \wedge (b \vee c)) \wedge (b \wedge c)\} \subseteq \{(a \wedge b) \wedge c\}$ .

If  $x_{c_{\gamma_1}} \in \{c^b\}$  then  $\gamma_1 \leq \beta_1$  because  $(x_{c_{\gamma_1}} \rightleftharpoons b_{\beta_1}) = 1$ . This means there is no  $b_{\beta_2} \in \{b\}$  with  $\alpha_1 \leq \beta_2 \leq \gamma_1$ ; i.e.,  $(b_{\beta_1} \rightleftharpoons y_{\alpha_1}) = 1$ , and hence  $(y_{\alpha_1} \rightleftharpoons b_{\beta_1})(b_{\beta_1} \rightleftharpoons c_{\gamma_1}) = 1$ ; i.e.,  $\{(a \wedge (b \vee c)) \wedge (b \wedge c)\} \subseteq \{(a \wedge b) \wedge c\}$ .

Let  $c_{\gamma_1} \in \{(a \wedge b) \wedge c\}$ , i.e.,  $c_{\gamma_1} \in \{c\}$  and there exists  $b_{\beta_1} \in \{a \wedge b\}$  such that  $(b_{\beta_1} \rightleftharpoons c_{\gamma_1}) = 1$ . Also for  $b_{\beta_1}$  there exists  $a_{\alpha_1} \in \{a\}$  such that  $(a_{\alpha_1} \rightleftharpoons b_{\beta_1}) = 1$ . Then  $c_{\gamma_1} \in \{b \wedge c\}$ , and since  $(b_{\beta_1} \rightleftharpoons c_{\gamma_1}) = 1$ , we may assume that  $a_{\alpha_1} \in \{a \wedge (b \vee c)\}$ , i.e.,  $\{(a \wedge b) \wedge c\} \subseteq \{(a \wedge (b \vee c)) \wedge (b \wedge c)\}$ .

$$8. (a \vee b) \vee ((a \wedge b) \vee c) = a \vee (b \vee c).$$

Is not hard to see that  $\{a \vee (b \vee c)\} = \{a\} \cup \{(b \vee c)^a\} = \{a\} \cup \{b^a\} \cup \{(c^b)^a\}$  because  $b \wedge c^b = e_\emptyset$ . Then we get  $\{(a \vee b) \vee ((a \wedge b) \vee c)\} = \{a\} \cup \{b^a\} \cup \{((a \wedge b) \vee c)^{(a \vee b)}\} = \{a\} \cup \{b^a\} \cup \{(a \wedge b)^{(a \vee b)}\} \cup \{c^{(a \vee b)}\} = \{a\} \cup \{b^a\} \cup \{c^{(a \vee b)}\}$ , but  $\{c^{(a \vee b)}\} = \{(c^b)^a\}$ ; i.e.,  $\{(a \vee b) \vee ((a \wedge b) \vee c)\} = \{a \vee (b \vee c)\}$ ; as claimed.

9.

$$a \vee b = b \iff a \wedge b = a$$

Indeed, since  $\{a \vee b\} := \{a\} \cup \{b^a\}$  and  $\{b\} = \{b_a\} \cup \{b^a\}$ , it follows from  $a \vee b = b$  that  $\{a\} = \{b_a\}$ .  $\square$

Suppose that  $a = (a_1; \dots; a_n) \in \mathbf{I}$  and  $\text{wt}(a) := (a_1 + \dots + a_n, a_1, \dots, a_n)$ . Order  $\mathbf{I}$  as follows: for any  $a, b \in \mathbf{I}$ , we say that  $a < b$  whenever  $\text{wt}(a) > \text{wt}(b)$  lexicographically. Then order  $\mathbf{I}^*$  by the deg-lex order.

**Remark 2.1.** Let  $a, b \in \mathbf{I}$ . Then Schensted's column algorithm and Definition 2.4 imply that  $a \cdot b$  is the leading term iff  $a \vee b \neq a$  and  $a \wedge b \neq b$ .

**Remark 2.2.** Put  $\mathcal{I} := \{a \cdot b = (a \vee b) \cdot (a \wedge b) : a, b \in \mathbf{I}\}$ . Then we may assume [5] that  $\mathbb{k}(\mathbf{I})/(\mathcal{I}) \cong \mathbb{k}(A)/(\Omega)$ . Then formulas (3.6), (3.7), (3.8) enable us to prove that  $\mathcal{I}$  is the Gröbner — Shirshov basis of the plactic monoid in column generators (see also [5, Theorem 4.3]). In fig. 5, we show a sketch of the Buchberger — Shirshov algorithm for the plactic monoid via the binary operations  $\vee$  and  $\wedge$ . In the knot theory spirit, we can interpret this operation as “overcrossing” and “undercrossing”.

**Remark 2.3.** The operations  $\vee, \wedge$  are not associative. Indeed, suppose that  $a = e_i, b = e_j$  and  $c = e_k$ . Assume that  $j < k < i$ . Then  $(e_i \vee e_j) \vee e_k = e_{ji} \vee e_k = e_{ji}e_k$ . But  $e_i \vee (e_j \vee e_k) = e_i \vee e_k = e_{ik}$ ; i.e.,

$$(a \vee b) \vee c \neq a \vee (b \vee c).$$

Assume now that  $j < i < k$ . Then  $(e_i \wedge e_j) \wedge e_k = e_\emptyset \wedge e_k = e_\emptyset$ . On other hand,  $e_i \wedge (e_j \wedge e_k) = e_i \wedge e_k = e_k$ ; i.e.,

$$(a \wedge b) \wedge c \neq a \wedge (b \wedge c).$$

However, we will use the notation  $a \vee (b \vee c) := a \vee b \vee c$  and  $(a \wedge b) \wedge c := a \wedge b \wedge c$ .

### 3 The Anick Resolution via Column Generators

Here we describe the Anick resolution for the  $\mathbb{k}\text{Pl}_n$ -module  $\mathbb{k}$  and for the  $\mathbb{k}\text{Pl}_n^e = \mathbb{k}\text{Pl}_n \otimes_{\mathbb{k}} \mathbb{k}\text{Pl}_n^\circ$ -module  $\mathbb{k}\text{Pl}_n$ .

**Lemma 3.1.** Given four arbitrary letters (columns)  $a, b, c, d$ , consider the word  $abcd$  and suppose that  $a \wedge b = b, b \wedge c \neq c$ , and  $c \wedge d = d$ . Then there is no reduction of this word to a word of the form  $a'b'c'd'$  such that  $a' \wedge b' \neq b', b' \wedge c' \neq c'$  or  $b' \wedge c' \neq c', c' \wedge d' \neq d'$ .

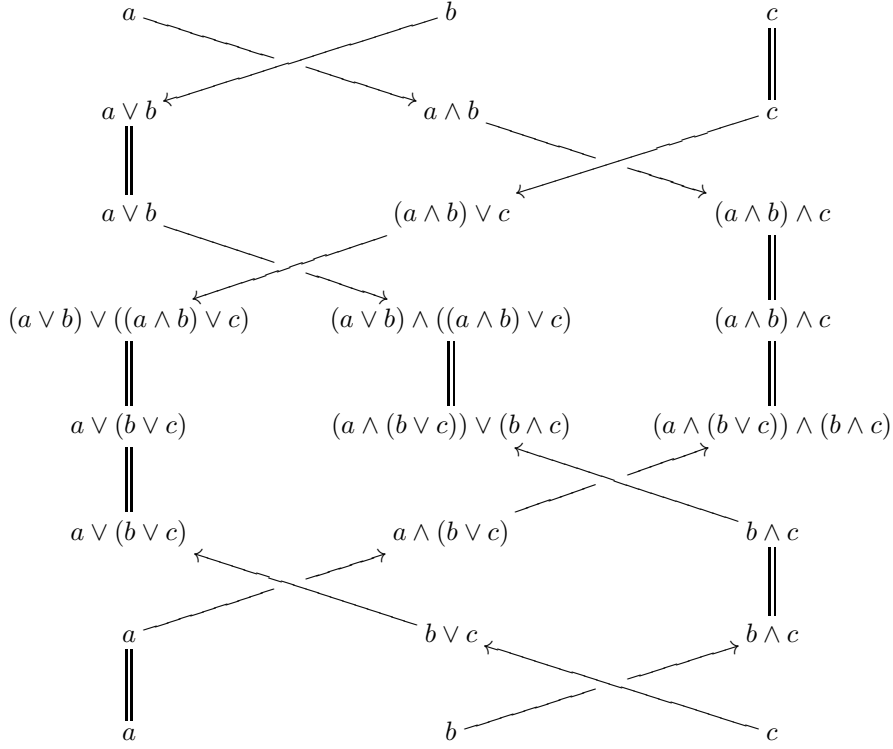
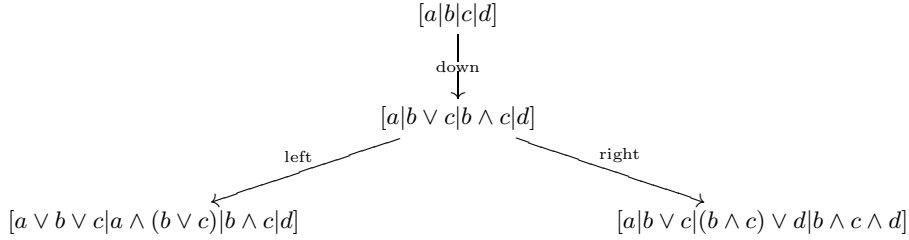


Figure 5: The “braid diagram” for the Buchberger — Shirshov algorithm for the plactic monoid via column generators.

*Proof.* We have



1. Consider the lower left side. Since  $a \vee b = a$ ,  $a \wedge b = b$ , (2.15) implies that

$$(a \wedge (b \vee c)) \vee (b \wedge c) = (a \vee b) \wedge ((a \wedge b) \vee c) = a \wedge (b \vee c);$$

i.e.,  $(a \wedge (b \vee c)) \cdot (b \wedge c)$  is not the leading term.

2. Consider the lower right side. Since  $c \vee d = c$ ,  $c \wedge d = d$ , (2.15) implies that

$$(b \vee c) \wedge ((b \wedge c) \vee d) = (b \wedge (c \vee d)) \vee (c \wedge d) = (b \wedge c) \vee d;$$

i.e.,  $(b \vee c) \cdot ((b \wedge c) \vee d)$  is not the leading term. □

**Lemma 3.2.** Let  $a$ ,  $b$ , and  $c$  be columns such that  $a \wedge b \neq b$  and  $b \wedge c \neq c$ . Then  $a \wedge (b \vee c) \neq b \vee c$  and  $(a \wedge b) \wedge c \neq c$ .

*Proof.* Theorem 2.1 implies that

$$a \wedge (b \vee c) \neq a \wedge b \neq b \neq b \vee c, \quad (a \wedge b) \wedge c \neq b \wedge c \neq c,$$

as claimed. □

**Theorem 3.1.** Let  $\mathbb{k}\text{Pl}(A)$  be the plactic monoid algebra over the field  $\mathbb{k}$  with augmentation  $\varepsilon : \mathbb{k}\text{Pl}(A) \rightarrow \mathbb{k}$ , and let  $I$  be a set of generators (columns) of the plactic monoid. Then the vector space  $\mathfrak{V}^{(m)}_{\mathbb{k}}$  spanned by the vectors  $a_1 \cdots a_{m+1}$  such that  $a_i \wedge a_{i+1} \neq a_{i+1}$  for all  $1 \leq i \leq m$  form an  $m$ -Anick chain; moreover, there is an (exact) chain complex of  $\mathbb{k}\text{Pl}_n$ -modules:

$$0 \leftarrow \mathbb{k} \xleftarrow{\varepsilon} \mathbb{k}\text{Pl}_n \xleftarrow{d_1} \mathbb{k}\text{Pl}_n \otimes_{\mathbb{k}} I \xleftarrow{d_2} \mathbb{k}\text{Pl}_n \otimes_{\mathbb{k}} \mathfrak{V} \xleftarrow{d_3} \mathbb{k}\text{Pl}_n \otimes_{\mathbb{k}} \mathfrak{V}^{(2)} \xleftarrow{\quad} \cdots,$$

where

$$d_n([a_1 | \dots | a_\ell]) = \sum_{i=0}^{\ell-1} (-1)^i (a_1 \vee \dots \vee a_{i+1}) [\widehat{L}_i] + \sum_{j=1}^{\ell} (-1)^j \varepsilon(a_i \wedge \dots \wedge a_\ell) [\widehat{R}_i] + \sum_{m=1}^{\ell-1} \sum_{m+l+k \leq \ell-1} (-1)^{l+k} W_{m,l,k}. \quad (3.18)$$

Here, for  $1 \leq i, j \leq \ell-1$ ,

$$[\widehat{L}_i] = \begin{cases} 0, & \text{iff } a_j \vee a_{j+1} = a_j \vee (a_{j+1} \vee a_{j+2}) \text{ for some } i \leq j \leq \ell-2, \\ [a_1 \wedge (a_2 \vee \dots \vee a_{i+1}) | \dots | a_{i-1} \wedge (a_i \vee a_{i+1}) | a_i \wedge a_{i+1} | a_{i+2} | \dots | a_\ell], & \text{otherwise.} \end{cases} \quad (3.19)$$

$$[\widehat{R}_i] = \begin{cases} 0, & \text{iff } a_j \wedge a_{j+1} = (a_j \wedge a_{j+1}) \wedge a_{j+2} \\ [a_1 | \dots | a_{i-1} | a_i \vee a_{i+1} | (a_i \wedge a_{i+1}) \vee a_{i+2} | \dots | (a_i \wedge \dots \wedge a_{\ell-1}) \vee a_n], & \text{otherwise.} \end{cases} \quad (3.20)$$

Here  $\widehat{L}_0 = [a_2 | \dots | a_\ell]$ ,  $\widehat{R}_\ell = [a_1 | \dots | a_\ell]$ ,

$$W_{m,l,k} = \begin{cases} [a_1 | \dots | a_{m-1} | b_m | \dots | b_{m+l-1} | b_{m+l} \vee c_{m+l+1} | c_{m+l+2} | \dots | c_{m+l+k} | a_{m+l+k+1} | \dots | a_\ell], & \text{iff } b_{m+l} \wedge c_{m+l+1} = 1_{\text{Pl}_n} \\ 0, & \text{otherwise,} \end{cases}$$

here

$$b_{m+\iota} = \begin{cases} a_m \vee a_{m+1}, & \text{if } \iota = 0 \\ (a_m \wedge \dots \wedge a_{m+\iota}) \vee a_{m+\iota+1}, & \text{if } 1 \leq \iota \leq l-1 \\ a_m \wedge \dots \wedge a_{m+l}, & \text{if } \iota = l \end{cases} \quad \text{or} \quad b_{m+\iota} = - \begin{cases} a_m \vee \dots \vee a_{m+l+1}, & \text{if } \iota = 0 \\ a_{m+\iota-1} \wedge (a_{m+\iota} \vee \dots \vee a_{m+l}), & \text{if } 1 \leq \iota \leq l-1 \\ a_{m+l-1} \wedge a_{m+l}, & \text{if } \iota = l \end{cases}$$

$$c_{m+\nu} = - \begin{cases} a_{m+l} \vee \dots \vee a_{m+l+k}, & \text{if } \nu = l+1 \\ a_{m+\nu-1} \wedge (a_{m+\nu} \vee \dots \vee a_{m+l+k}), & \text{if } \nu = l+t, 1 \leq t < k \\ a_{m+l+k-1} \wedge a_{m+l+k}, & \text{if } \nu = l+k \end{cases} \quad \text{or} \quad c_{m+\nu} = \begin{cases} a_{m+l+1} \vee a_{m+l+2}, & \text{if } \nu = l+1 \\ (a_{m+l+1} \wedge \dots \wedge a_{m+\nu}) \vee a_{m+\nu+1}, & \text{if } \nu = l+t, 1 < t < k \\ a_{m+l+1} \wedge \dots \wedge a_{m+l+k}, & \text{if } \nu = l+k. \end{cases}$$

*Proof.* Since the Gröbner — Shirshov basis of the plactic monoid via column generators is quadratic non-homogeneous,  $\mathfrak{V}^{(m)} = \{a_1 \dots a_{m+1} : a_i \wedge a_{i+1} \neq a_{i+1}, \text{ for all } 1 \leq i \leq m\}$ , for any  $m > 1$ . Following [21], we will use the bar notation  $[a_1 | \dots | a_{\ell+1}]$  for an  $\ell$ th Anick chain.

Let  $[a_1 | \dots | a_\ell] \in \mathfrak{V}^{(\ell-1)}$  be an  $\ell-1$ -Anick chain. Theorem 1.2 and Proposition 1.1 tell us that first we must find all weighted paths  $p_i : [a_1 | \dots | a_\ell] \xrightarrow{\omega_i} [b_1 | \dots | b_{\ell-1}]$  such that  $[b_1 | \dots | b_{\ell-1}] \in \mathfrak{V}^{(\ell-2)}$ .

We say that the  $n$ -tuple  $[a_1 | \dots | a_i \vee a_{i+1} | a_i \wedge a_{i+1} | \dots | a_\ell]$  has a hole at the point  $i$ . Lemma 3.2 implies that we can move this hole to the left or to the right in the following sense:

$$[a_1 | \dots | a_i \vee a_{i+1} | a_i \wedge a_{i+1} | \dots | a_\ell] \rightarrow [a_1 | \dots | a_i \vee a_{i+1} | (a_i \wedge a_{i+1}) \vee a_{i+2} | (a_i \wedge a_{i+1}) \wedge a_{i+2} | \dots | a_\ell] \quad \text{movement of the hole to the right by one step}$$

$$\begin{aligned} & \text{movement of the hole to the left by one step} \quad [a_1 | \dots | a_i \vee a_{i+1} | a_i \wedge a_{i+1} | \dots | a_\ell] \rightarrow \\ & \rightarrow [a_1 | \dots | a_{i-1} \vee (a_i \vee a_{i+1}) | a_{i-1} \wedge (a_i \vee a_{i+1}) | a_i \wedge a_{i+1} | \dots | a_\ell]. \end{aligned}$$

Lemma 3.1 implies that, for finding paths  $p_i : [a_1 | \dots | a_\ell] \xrightarrow{\omega_i} [b_1 | \dots | b_{\ell-1}]$ , where  $[b_1 | \dots | b_{\ell-1}] \in \mathfrak{V}^{(\ell-2)}$ , we cannot make more than one hole in the tuple  $[a_1 | \dots | a_\ell]$ . Assume that  $(a_i \vee a_{i+1}) \vee ((a_i \wedge a_{i+1}) \vee a_{i+2}) \neq a_i \vee a_{i+1}$  and  $(a_i \wedge (a_{i+1} \vee a_{i+2})) \wedge (a_{i+1} \wedge a_{i+2}) \neq a_{i+1} \wedge a_{i+2}$  for any  $1 \leq i \leq \ell-2$ . Then all paths  $p_i$  have the form

$$L_i : [a_1 | \dots | a_\ell] \rightarrow [a_1 \vee \dots \vee a_{i+1} | a_1 \wedge (a_2 \vee \dots \vee a_{i+1}) | a_2 \wedge (a_3 \vee \dots \vee a_{i+1}) | \dots | a_{i-1} \wedge (a_i \vee a_{i+1}) | a_i \wedge a_{i+1} | a_{i+2} | \dots | a_\ell],$$

$$R_i : [a_1 | \dots | a_\ell] \rightarrow [a_1 | \dots | a_{i-1} | a_i \vee a_{i+1} | (a_i \wedge a_{i+1}) \vee a_{i+2} | (a_i \wedge a_{i+1} \wedge a_{i+2}) \vee a_{i+3} | \dots | (a_i \wedge \dots \wedge a_{n-1}) \vee a_\ell | a_1 \wedge \dots \wedge a_\ell].$$

Since  $\Gamma([a_1 | \dots | a_\ell] \rightarrow L_i) = (-1)^i$  and  $\Gamma([a_1 | \dots | a_\ell] \rightarrow R_i) = (-1)^{\ell-i}$ , (1.3) implies

$$\Gamma([a_1 | \dots | a_\ell] \rightarrow L_i \rightarrow \widehat{L_i}) = (-1)^i \varepsilon(a_1 \vee \dots \vee a_{i+1}), \quad \Gamma([a_1 | \dots | a_\ell] \rightarrow R_i \rightarrow \widehat{R_i}) = (-1)^i (a_i \wedge \dots \wedge a_\ell),$$

and Proposition 1.1 yields (3.18).

Now, suppose that, for some  $0 \leq i \leq \ell$ , we have  $(a_i \vee a_{i+1}) \wedge ((a_i \wedge a_{i+1}) \vee a_{i+2}) = ((a_i \wedge a_{i+1}) \vee a_{i+2})$  or  $(a_i \wedge (a_{i+1} \vee a_{i+2})) \wedge (a_{i+1} \wedge a_{i+2}) = a_{i+1} \wedge a_{i+2}$ . Theorem 2.1 implies that

$$a_i \vee a_{i+1} = (a_i \vee a_{i+1}) \vee ((a_i \wedge a_{i+1}) \vee a_{i+2}) = a_i \vee (a_{i+1} \vee a_{i+2}),$$

$$a_{i+1} \wedge a_{i+2} = (a_i \wedge (a_{i+1} \vee a_{i+2})) \wedge (a_{i+1} \wedge a_{i+2}) = (a_i \wedge a_{i+1}) \wedge a_{i+2},$$

and we must put  $\widehat{R_i} = 0$  (respectively,  $\widehat{L_i} = 0$ ). Finally, assuming that there are equalities of the form  $a' \wedge b' = 1_{\text{Pl}_n}$ , we infer that there are paths of the form  $W_{m,l,k}$  with weight  $(-1)^{l+k}$ . This completes the proof.  $\square$

**Theorem 3.2.** *In the above notation, we obtain the (exact) chain complex of  $\mathbb{K}\text{Pl}_n^e$ -modules*

$$0 \leftarrow \mathbb{K}\text{Pl}_n^e \xleftarrow{d_0} \mathbb{K}\text{Pl}_n^e \otimes_{\mathbb{K}} \mathbb{K} \xleftarrow{d_1} \mathbb{K}\text{Pl}_n^e \otimes_{\mathbb{K}} \mathfrak{V} \xleftarrow{d_2} \mathbb{K}\text{Pl}_n^e \otimes_{\mathbb{K}} \mathfrak{V}^{(2)} \xleftarrow{d_3} \dots,$$

where

$$\begin{aligned} d_n([a_1 | \dots | a_\ell]) &= \\ &= \sum_{i=0}^{\ell-1} (-1)^i ((a_1 \vee \dots \vee a_{i+1}) \otimes 1) [\widehat{L_i}] + \sum_{j=1}^{\ell} (-1)^j (1 \otimes (a_j \wedge \dots \wedge a_\ell)) [\widehat{R_j}] + \sum_{m=1}^{\ell-1} \sum_{m+l+k \leq \ell-1} (-1)^{l+k} W_{m,l,k}. \end{aligned} \quad (3.21)$$

*Proof.* The proof is the same as that of Theorem 3.1 with the exception of weights. As in the proof of Theorem 3.1, we can give an explicit description of the paths:

$$L_i : [a_1 | \dots | a_\ell] \rightarrow [a_1 \vee \dots \vee a_{i+1} | a_1 \wedge (a_2 \vee \dots \vee a_{i+1}) | a_2 \wedge (a_3 \vee \dots \vee a_{i+1}) | \dots | a_{i-1} \wedge (a_i \vee a_{i+1}) | a_i \wedge a_{i+1} | a_{i+2} | \dots | a_\ell],$$

$$R_i : [a_1 | \dots | a_\ell] \rightarrow [a_1 | \dots | a_{i-1} | a_i \vee a_{i+1} | (a_i \wedge a_{i+1}) \vee a_{i+2} | (a_i \wedge a_{i+1} \wedge a_{i+2}) \vee a_{i+3} | \dots | (a_i \wedge \dots \wedge a_{n-1}) \vee a_\ell | a_1 \wedge \dots \wedge a_\ell].$$

It follows from (1.4)

$$\Gamma([a_1 | \dots | a_\ell] \rightarrow L_i \rightarrow \widehat{L_i}) = (-1)^i (a_1 \vee \dots \vee a_{i+1}) \otimes 1, \quad \Gamma([a_1 | \dots | a_\ell] \rightarrow R_i \rightarrow \widehat{R_i}) = (-1)^i 1 \otimes (a_i \wedge \dots \wedge a_\ell),$$

and Proposition 1.2 gives (3.21).  $\square$

## 4 The Cohomology Ring of the Plactic Monoid Algebra

We will use the notations  $\widehat{L_i}[a_1 | \dots | a_\ell] = \widehat{L_i} = (a_1 \wedge (a_2 \vee \dots \vee a_{i+1})) \dots (a_{i-1} \wedge (a_i \vee a_{i+1})) (a_i \wedge a_{i+1}) (a_{i+2} \vee \dots \vee a_\ell)$  and  $\widehat{R_i}[a_1 | \dots | a_\ell] = \widehat{R_i} = ((a_1 \vee \dots \vee a_{i+1}) \wedge (a_i \wedge a_{i+1})) \vee a_{i+2} \vee \dots \vee a_\ell$ . Here  $0 \leq i \leq \ell-1$  and  $1 \leq j \leq \ell$ .

**Lemma 4.1.** *Let  $M$  be a  $\mathbb{K}\text{Pl}_n$ -module and let  $\xi \in \text{Hom}_{\mathbb{K}}(\mathbb{K}, M)$ ,  $\zeta \in \text{Hom}_{\mathbb{K}}(\mathfrak{V}^{(\ell-1)}, M)$ . Then  $\xi \smile \zeta, \zeta \smile \xi \in \text{Hom}_{\mathbb{K}}(\mathfrak{V}^{(\ell)}, M)$  can be described by the formulas*

$$(\xi \smile \zeta)[a_1 | \dots | a_{\ell+1}] = \sum_{i=0}^{\ell} (-1)^i \left( \xi[a_1 \vee \dots \vee a_{i+1}] \varepsilon(\widehat{L_i}) \right) \left( (a_1 \vee \dots \vee a_{i+1}) \zeta[\widehat{L_i}] \right), \quad (4.22)$$

$$(\zeta \smile \xi)[a_1 | \dots | a_{\ell+1}] = \sum_{j=1}^{\ell+1} (-1)^{\ell+1-j} \left( \zeta[\widehat{R_j}] \varepsilon(a_j \wedge \dots \wedge a_{\ell+1}) \right) \left( \widehat{R_j} \xi[a_j \wedge \dots \wedge a_{\ell+1}] \right). \quad (4.23)$$

*Proof.* Indeed, from (1.7) follows that we need to find all paths  $\{p\}$  of forms,

$$\mathfrak{V}^{(\ell)} \ni [a_1 | \dots | a_{\ell+1}] \rightarrow [b_1 | \dots | b_\ell] \in \mathfrak{V}^{(\ell-1)}$$

but from construction of  $\widehat{R}, \widehat{L}$  (see (3.19), (3.20)) follows that  $\{p\} = \{\widehat{R}, \widehat{L}\}$ , and weights of this paths were found in proof of Theorem 3.1. This completes the proof.  $\square$



*Proof.* Since there are no relations of the form  $ab = \alpha \in \mathbb{k}$ , we may assume that the augmentation map  $\varepsilon : \mathbb{k}\text{Pl}_n \rightarrow \mathbb{k}$  is the identity map, i.e.,  $\varepsilon(x) = 1$  for any  $x \in \text{Pl}_n$ . We get the cochain complex

$$0 \rightarrow \mathbb{k}\text{Pl}_n \xrightarrow{d^0} \text{Hom}_{\mathbb{k}}(\mathbb{I}\mathbb{k}, \mathbb{k}) \xrightarrow{d^1} \text{Hom}_{\mathbb{k}}(\mathfrak{V}\mathbb{k}, \mathbb{k}) \xrightarrow{d^2} \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(2)}\mathbb{k}, \mathbb{k}) \rightarrow \dots$$

where

$$(d^0 x)[a] = \varepsilon(a) - \varepsilon(a) = 0, \quad (d^1 \xi)[a|b] = \xi[a] + \xi[b] - \xi[a \vee b] - \xi[a \wedge b],$$

and for any  $\varphi \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(\ell-1)}\mathbb{k}, \mathbb{k})$  we have

$$(d^\ell \varphi)[a_1 | \dots | a_{\ell+1}] = \sum_{i=0}^{\ell} (-1)^i \varphi[\widehat{L_i}] + \sum_{j=1}^{\ell+1} (-1)^j \varphi[\widehat{R_j}] + \sum_{m=1}^{\ell-1} \sum_{m+l+k \leq \ell-1} (-1)^{l+k} \varphi(W_{m,l,k}).$$

Consider the following functions:

$$\xi_i(x) = \begin{cases} 1, & \text{if } \{e_i\} \subseteq \{x\}, \\ 0, & \text{otherwise,} \end{cases}$$

here  $e_i, x = e_{x_1, \dots, x_\ell} \in \mathbb{I}$ .

It's not hard to see that  $\xi_i(x)$  are cocycles. Indeed, there are following possibilities, 1) let  $\{e_i\} \subseteq \{a\}$  and  $\{e_i\} \in \{b\}$  then  $\{e_i\} \in \{a \vee b\}$  and  $\{e_i\} \in \{a \wedge b\}$  it follows  $(d^1 \xi_i)[a|b] = 1 + 1 - 1 - 1 = 0$ , 2) let  $\{e_i\} \in \{a\}$  and  $\{e_i\} \notin \{b\}$  then  $\{e_i\} \in \{a \vee b\}$  and  $\{e_i\} \notin \{a \wedge b\}$  it follows  $(d^1 \xi_i)[a|b] = 1 + 0 - 1 - 0 = 0$ , 3) let  $\{e_i\} \notin \{a\}$  and  $\{e_i\} \in \{b\}$  then we have to consider two cases; 3a)  $\{e_i\} \in \{a \vee b\}$  then  $\{e_i\} \notin \{a \wedge b\}$  it follows  $(d^1 \xi_i)[a|b] = 0 + 1 - 1 - 0 = 0$ , 3b)  $\{e_i\} \notin \{a \vee b\}$  then  $\{e_i\} \in \{a \wedge b\}$  it follows  $(d^1 \xi_i)[a|b] = 0 + 1 - 0 - 1 = 0$ , 4) let  $\{e_i\} \notin \{a\}$  and  $\{e_i\} \notin \{b\}$  then  $\{e_i\} \notin \{a \vee b\}$  and  $\{e_i\} \notin \{a \wedge b\}$  it follows  $(d^1 \xi_i)[a|b] = 0 + 0 - 0 - 0 = 0$ .

Let  $\vartheta_p \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p-1)}\mathbb{k}, \mathbb{k})$  and let  $\vartheta_q \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(q-1)}\mathbb{k}, \mathbb{k})$ . Since the comultiplication  $\mathbb{k}\text{Pl}_n \otimes \mathbb{k}\text{Pl}_n \leftarrow \mathbb{k}\text{Pl}_n : \Delta(x) = x \otimes x$  is cocommutative, then the product  $\smile$  must be skew commutative, i.e.  $(\vartheta_p \smile \vartheta_q) = (-1)^{pq}(\vartheta_q \smile \vartheta_p)$ . Consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p-1)}\mathbb{k}, \mathbb{k}) \otimes \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(q-1)}\mathbb{k}, \mathbb{k}) & \xrightarrow{\smile} & \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p+q-1)}\mathbb{k}, \mathbb{k}) \\ \tau^* \downarrow & & \downarrow \tilde{\tau} \\ \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(q-1)}\mathbb{k}, \mathbb{k}) \otimes \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p-1)}\mathbb{k}, \mathbb{k}) & \xrightarrow{\smile} & \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p+q-1)}\mathbb{k}, \mathbb{k}). \end{array}$$

Here  $\tau : A_{\bullet}(\mathbb{k}\text{Pl}_n, \mathbb{k}) \otimes A_{\bullet}(\mathbb{k}\text{Pl}_n, \mathbb{k}) \rightarrow A_{\bullet}(\mathbb{k}\text{Pl}_n, \mathbb{k}) \otimes A_{\bullet}(\mathbb{k}\text{Pl}_n, \mathbb{k})$  is the chain automorphism such that  $\tau : x \otimes y \rightarrow (-1)^{\deg(x)\deg(y)} y \otimes x$ .

We may assume without loss of generality that  $p = 1$  and  $q = \ell$ . Suppose that  $\xi \in \text{Hom}_{\mathbb{k}}(\mathbb{I}\mathbb{k}, \mathbb{k})$  and  $\zeta \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(\ell-1)}\mathbb{k}, \mathbb{k})$ . Then  $\xi \smile \zeta = (-1)^{\ell} \zeta \smile \xi$ . Using (4.22), we obtain

$$\begin{aligned} (\xi \smile \zeta)[a_1 | \dots | a_{\ell+1}] &\rightsquigarrow (\xi \vee \zeta) \left( \sum_{i=0}^{\ell} (-1)^i [a_1 \vee \dots \vee a_{i+1}] \otimes [\widehat{L_i}] \right) \xrightarrow{\tau^*} \\ &\xrightarrow{\tau^*} (-1)^{\ell} (\zeta \vee \xi) \left( \sum_{i=0}^{\ell} (-1)^i \widehat{L_i} \otimes [a_1 \vee \dots \vee a_{i+1}] \right) \rightsquigarrow (-1)^{\ell} (\zeta \smile \xi)[a_1 | \dots | a_{\ell+1}], \end{aligned}$$

and (4.23) implies that

$$\sum_{j=1}^{\ell+1} (-1)^j [\widehat{R_j}] \otimes [a_1 \wedge \dots \wedge a_{\ell+1}] = \sum_{i=0}^{\ell} (-1)^{i+1} [\widehat{L_i}] \otimes [a_1 \vee \dots \vee a_{i+1}].$$

Lemma 4.3 now gives that if  $ab = ba$  for all  $a, b \in \{a_1, \dots, a_{\ell+1}\}$  then  $(\xi \smile \zeta)[a_1 | \dots | a_{\ell+1}] \neq 0$ . Finally, using Corollary 4.0.1 and (4.22), we have

$$\begin{aligned} (\vartheta_p \smile \vartheta_q)[a_1 | \dots | a_{p+q}] &= \\ &= \begin{cases} \sum_{\substack{1 \leq i_1 < \dots < i_p \leq p \\ 1 \leq j_1 < \dots < j_q \leq q}} \rho_{PQ} \vartheta_p[a_{i_1} | \dots | a_{i_p}] \vartheta_q[a_{j_1} | \dots | a_{j_q}], & \text{iff } a_i a_{i+1} = a_{i+1} a_i \text{ for all } 1 \leq i \leq p+q-1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (4.26)$$

Let us assume now all columns  $a_1, \dots, a_{\ell}$  are pairwise commutative, then from (4.26) follows that

$$\xi_{i_1} \smile \dots \smile \xi_{i_{\ell}}[a_1 | \dots | a_{\ell}] = \left\| \begin{array}{ccc} \xi_{i_1}(a_1) & \dots & \xi_{i_1}(a_{\ell}) \\ \vdots & \ddots & \vdots \\ \xi_{i_{\ell}}(a_1) & \dots & \xi_{i_{\ell}}(a_{\ell}) \end{array} \right\| \quad (4.27)$$

It's not hard to see that if  $\xi_{i_k}(a_j) = 0$  then  $\xi_{i_k}(a_{j-t}) = 0$  for any  $1 \leq t \leq j-1$ , then using Laplace expansion we can express the determinant via determinants of sub-matrices. It means our determinant is non zero iff the correspondence matrix is upper triangular matrix, q.e.d.  $\square$

**The Hochschild Cohomology Ring  $HH^*(\mathbb{P}l_n)$  of the plactic monoid algebra.** Theorem 3.2 implies that the Hochschild cohomology of the plactic monoid algebra is isomorphic to the homology of the cochain complex

$$0 \rightarrow \mathbb{K}\mathbb{P}l_n \xrightarrow{d^0} \text{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K}\mathbb{P}l_n) \xrightarrow{d^1} \text{Hom}_{\mathbb{K}}(\mathfrak{V}\mathbb{K}, \mathbb{K}\mathbb{P}l_n) \xrightarrow{d^2} \text{Hom}_{\mathbb{K}}(\mathfrak{V}^{(2)}\mathbb{K}, \mathbb{K}\mathbb{P}l_n) \rightarrow \dots$$

Here, for  $w \in \mathbb{K}\mathbb{P}l_n$ ,  $a, b, a_1, \dots, a_n \in I$  and  $\psi \in \text{Hom}_{\mathbb{K}}(\mathfrak{V}^{(\ell-1)}, \mathbb{K}\mathbb{P}l_n)$ , we have

$$(d^0 w)[a] = f(wa) - f(aw), \quad (d^1 \psi^1)[a|b] = \psi[a]b + a\psi[b] - \psi[a \vee b](a \wedge b) - (a \vee b)\psi[a \wedge b],$$

$$\begin{aligned} (d^\ell \psi)[a_1 | \dots | a_{\ell+1}] &= \\ &= \sum_{i=0}^{\ell} (-1)^i (a_1 \vee \dots \vee a_{i+1}) \psi[\widehat{L_i}] + \sum_{j=1}^{\ell+1} (-1)^j \psi[\widehat{R_j}](a_j \wedge \dots \wedge a_m) + \sum_{m=1}^{\ell-1} \sum_{m+l+k \leq \ell-1} (-1)^{l+k} \psi(W_{m,l,k}). \end{aligned}$$

As is well known [9, Lemma 3] the center  $Z(\mathbb{P}l_n)$  is equal to the cyclic monoid  $\langle e_1, \dots, e_n \rangle$ , i.e.,  $HH^0(\mathbb{K}\mathbb{P}l_n) \cong \mathbb{K}[e_1, \dots, e_n]$ .

**Proposition 4.1.** *For any columns  $e_i, a \in I$ , the cochains  $\frac{\partial a}{\partial e_i} : \mathbb{K} \rightarrow \mathbb{K}\mathbb{P}l_n$  defined by the rule*

$$\frac{\partial a}{\partial e_i} = \begin{cases} \{a \setminus e_i\}, & \text{if } \{e_i\} \subseteq \{a\}, \\ 0, & \text{otherwise.} \end{cases}$$

are derivations, moreover these derivations are not inner.

*Proof.* For any column  $c = (c_1; \dots; c_n)$ , put  $|c| = c_1 + \dots + c_n$ . Let us prove that  $\frac{\partial}{\partial e_i} \notin \text{Im}(d^0)$ . Assume that  $\lambda \vee a = \frac{\partial a}{\partial e_i}$  for some  $\lambda \in I$ . Then (2.12) implies that  $a = \frac{\partial a}{\partial e_i} \wedge a$ , but then  $\left| \frac{\partial a}{\partial e_i} \right| \leq |a|$  leads to a contradiction. Suppose now that  $a \vee \lambda = \frac{\partial a}{\partial e_i}$ . Then  $|a \vee \lambda| \geq |a|$  gives a contraction. This means that  $\frac{\partial}{\partial e_i} \notin \text{Im}(d^0)$ .

Show that this is a one-dimensional cocycle. We infer

$$\left( d^1 \frac{\partial}{\partial e_i} \right) [a|b] = \frac{\partial a}{\partial e_i} b + a \frac{\partial b}{\partial e_i} - \frac{\partial(a \vee b)}{\partial e_i} (a \wedge b) - (a \vee b) \frac{\partial(a \wedge b)}{\partial e_i}.$$

We can represent  $\frac{\partial y}{\partial e_i}$  as  $\left\{ \frac{\partial y}{\partial e_i} \right\} = \delta_{\{e_i\} \subseteq \{y\}} \{y\} \setminus \{e_i\}$ , where

$$\delta_{\{e_i\} \subseteq \{y\}} = \begin{cases} 1, & \text{iff } \{e_i\} \subseteq \{y\}, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain:

$$\begin{aligned} \left( d^1 \frac{\partial}{\partial e_i} \right) [a|b] &= \delta_{\{e_i\} \subseteq \{a\}} (\{a\} \setminus \{e_i\})b + \delta_{\{e_i\} \subseteq \{b\}} a(\{b\} \setminus \{e_i\}) - \\ &\quad - \delta_{\{e_i\} \subseteq \{a \vee b\}} (\{(a \vee b)\} \setminus \{e_i\})(a \wedge b) - \delta_{\{e_i\} \subseteq \{a \wedge b\}} (a \vee b)(\{a \wedge b\} \setminus \{e_i\}). \end{aligned}$$

Case 1.  $\delta_{\{e_i\} \subseteq \{b\}} = \delta_{\{e_i\} \subseteq \{a \wedge b\}} = 0$ ,  $\delta_{\{e_i\} \subseteq \{a\}} = \delta_{\{e_i\} \subseteq \{a \vee b\}} = 1$ . Then  $\{(a \setminus e_i) \vee b\} = \{(a \setminus e_i)\} \cup \{b^a\}$ , and  $\{(a \setminus e_i) \wedge b\} = \{b_a\}$ , and also we see that  $\{(a \vee b) \setminus e_i\} = \{a \setminus e_i\} \cup \{b^a\}$ .

Case 2.  $\delta_{\{e_i\} \subseteq \{b\}} = \delta_{\{e_i\} \subseteq \{a \wedge b\}} = 1$  and  $\delta_{\{e_i\} \subseteq \{a\}} = \delta_{\{e_i\} \subseteq \{a \vee b\}} = 0$ . Then  $\{(a \wedge b) \setminus e_i\} = \{a \wedge (b \setminus e_i)\}$  and  $\{a \vee (b \setminus e_i)\} = \{a\} \cup \{b^a\}$ . It is also not hard to see that  $\{((a \wedge b) \setminus e_i)^{a \vee b}\} = \{(a \wedge b) \setminus e_i\} = \{a \wedge (b \setminus e_i)\}$ .

Case 3. Suppose that  $\delta_{\{e_i\} \subseteq \{a \wedge b\}} = 0$  and  $\delta_{\{e_i\} \subseteq \{b\}} = 1$ . Then  $\delta_{\{e_i\} \subseteq \{a\}} = 0$ . We get  $\{(a \vee b) \setminus e_i\} = \{a\} \cup \{b^a \setminus e_i\} = \{a\} \cup \{(b \setminus e_i)^a\}$ , and  $\{a \wedge (b \setminus e_i)\} = \{a \wedge b\}$ .

Case 4.  $\delta_{\{e_i\} \subseteq \{b\}} = \delta_{\{e_i\} \subseteq \{a \wedge b\}} = 1$  and  $\delta_{\{e_i\} \subseteq \{a\}} = \delta_{\{e_i\} \subseteq \{a \vee b\}} = 1$ . Then  $\{e_i\} \subseteq \{a\} \cap \{b\}$ . We see that  $\{(a \vee b) \setminus e_i\} = \{a \setminus e_i\} \cup \{b^a\}$  and  $\{(a \setminus e_i) \wedge b\} = \{(a \wedge b) \setminus e_i\}$ . Therefore,  $\{((a \vee b) \setminus e_i) \wedge (a \wedge b)\} = \{((a \setminus e_i) \vee b) \wedge (a \wedge b)\} = \{((a \vee b) \wedge (a \wedge b)) \setminus e_i\} = \{(a \wedge b) \setminus e_i\}$ . Further,  $\{a \wedge (b \setminus e_i)\} = \{(a \wedge b) \setminus e_i\}$  and  $\{(a \vee b) \vee ((a \wedge b) \setminus e_i)\} = \{((a \vee b) \vee (a \wedge b)) \setminus e_i\} = (a \vee b) \setminus e_i$ .

If we assume that  $\delta_{\{e_i\} \subseteq \{a \vee b\}} = 1$  then  $\delta_{\{e_i\} \subseteq \{a\}} = 1$ ; otherwise, if  $\delta_{\{e_i\} \subseteq \{a \vee b\}} = 0$  then  $\delta_{\{e_i\} \subseteq \{a\}} = 0$ . This means that all possible cases are considered. Moreover any column  $a \in I$  which contains more than one element, i.e.,  $|c| > 1$  can be present as  $c = e_i \cdot c'$  where  $e_i$  corresponds to the minimal element of column  $c$ , it means that the cochains  $\frac{\partial}{\partial e_i}$  are generators of  $HH^1(\mathbb{K}\mathbb{P}l_n)$ . □

**Proposition 4.2.** *Let  $\mathbb{k}\text{Pl}_n$  be the plactic monoid algebra over a field. Then the multiplication in Hochschild cohomology ring  $HH^*(\mathbb{k}\text{Pl}_n)$  can be described as follows:*

$$\begin{aligned} \left( \frac{\partial}{\partial e_{i_1}} \smile \cdots \smile \frac{\partial}{\partial e_{i_\ell}} \right) [a_1 | \dots | a_\ell] &= \\ &= \begin{cases} \sum_{\sigma \in \mathbb{S}_n} \text{sign}(\sigma) (a_\ell \cdots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \cdots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}}, & \text{iff } a_i a_j = a_j a_i \text{ for all } 1 \leq i, j \leq \ell, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (4.28)$$

here  $\text{sign}(\sigma) = \text{sign} \begin{pmatrix} 1 & \cdots & \ell \\ \sigma(1) & \cdots & \sigma(\ell) \end{pmatrix}$

*Proof.* Let  $\psi_p \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n)$  and let  $\psi_q \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(q-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n)$ . Since the comultiplication  $\mathbb{k}\text{Pl}_n \otimes \mathbb{k}\text{Pl}_n \leftarrow \mathbb{k}\text{Pl}_n : \Delta(x) = x \otimes x$  is cocommutative, the product  $\smile$  must be skew commutative; i.e.,  $(\psi_p \smile \psi_q) = (-1)^{pq}(\psi_q \smile \psi_p)$ . Consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n) \otimes \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(q-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n) & \xrightarrow{\smile} & \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p+q-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n) \\ \tau^* \downarrow & & \downarrow \check{\tau} \\ \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(q-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n) \otimes \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n) & \xrightarrow{\smile} & \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(p+q-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n). \end{array}$$

Here  $\tau : A_\bullet(\mathbb{k}\text{Pl}_n, \mathbb{k}\text{Pl}_n) \otimes A_\bullet(\mathbb{k}\text{Pl}_n, \mathbb{k}\text{Pl}_n) \rightarrow A_\bullet(\mathbb{k}\text{Pl}_n, \mathbb{k}\text{Pl}_n) \otimes A_\bullet(\mathbb{k}\text{Pl}_n, \mathbb{k}\text{Pl}_n)$  is the chain automorphism such that  $\tau : x \otimes y \rightarrow (-1)^{\deg(x)\deg(y)} y \otimes x$ .

We may assume without loss of generality that  $p = 1$  and  $q = \ell$ . Suppose that  $\alpha \in \text{Hom}_{\mathbb{k}}(\mathbb{k}, \mathbb{k}\text{Pl}_n)$  and  $\beta \in \text{Hom}_{\mathbb{k}}(\mathfrak{V}^{(\ell-1)}\mathbb{k}, \mathbb{k}\text{Pl}_n)$ . Then we have  $\alpha \smile \beta = (-1)^\ell \beta \smile \alpha$ . Using (4.24), we get

$$\begin{aligned} (\alpha \smile \beta)[a_1 | \dots | a_{\ell+1}] &\rightsquigarrow (\alpha \vee \beta) \left( \sum_{i=0}^{\ell} (-1)^i \left( \widehat{L}_i \otimes (a_1 \vee \dots \vee a_{i+1}) \right) [a_1 \vee \dots \vee a_{i+1}] \otimes [\widehat{L}_i] \right) \xrightarrow{\tau^*} \\ &\xrightarrow{\tau^*} (-1)^\ell (\beta \vee \alpha) \left( \sum_{i=0}^{\ell} (-1)^i \left( \widehat{L}_i \otimes (a_1 \vee \dots \vee a_{i+1}) \right) [\widehat{L}_i] \otimes [a_1 \vee \dots \vee a_{i+1}] \right) \rightsquigarrow (-1)^\ell (\beta \smile \alpha)[a_1 | \dots | a_{\ell+1}] \end{aligned}$$

Now, it follows from (4.25) that

$$\sum_{j=1}^{\ell+1} (-1)^j \widehat{R}_j \otimes [a_1 \wedge \dots \wedge a_{\ell+1}] = \sum_{i=0}^{\ell} (-1)^{i+1} \widehat{L}_i \otimes [a_1 \vee \dots \vee a_{i+1}].$$

Using Lemma 4.3, we obtain that if  $ab = ba$  for all  $a, b \in \{a_1, \dots, a_{\ell+1}\}$  then  $(\alpha \smile \beta)[a_1 | \dots | a_{\ell+1}] \neq 0$ . Finally, making use of Corollary 4.0.1 and (4.24), we obtain

$$\begin{aligned} (\psi_p \smile \psi_q)[a_1 | \dots | a_{p+q}] &= \\ &= \begin{cases} \sum_{\substack{1 \leq i_1 < \dots < i_p \leq p \\ 1 \leq j_1 < \dots < j_q \leq q}} \rho_{PQ} \psi_p[a_{i_1} | \dots | a_{i_p}] (a_1 \cdots a_{p+q}) \psi_q[a_{j_1} | \dots | a_{j_q}] & \text{if } a_i a_{i+1} = a_{i+1} a_i \text{ for all } 1 \leq i \leq p+q-1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This means that

$$\frac{\partial^2}{\partial x \partial y} [a|b] := \left( \frac{\partial}{\partial x} \smile \frac{\partial}{\partial y} \right) [a|b] = \left( \frac{\partial a}{\partial x} b \right) \left( a \frac{\partial b}{\partial y} \right) - \left( \frac{\partial(a \vee b)}{\partial x} (a \wedge b) \right) \left( (a \vee b) \frac{\partial(a \wedge b)}{\partial y} \right),$$

and, more generally,

$$\begin{aligned} \frac{\partial^{p+q}}{\partial x_1 \cdots \partial x_p \partial y_1 \cdots \partial y_q} [a_1 | \dots | a_{p+q}] &:= \left( \frac{\partial^p}{\partial x_1 \cdots \partial x_p} \smile \frac{\partial^q}{\partial y_1 \cdots \partial y_q} \right) [a_1 | \dots | a_{p+q}] = \\ &= \begin{cases} \sum_{\substack{1 \leq i_1 < \dots < i_p \leq p \\ 1 \leq j_1 < \dots < j_q \leq q}} \rho_{PQ} \left( \frac{\partial^p}{\partial x_1 \cdots \partial x_p} [a_{i_1} | \dots | a_{i_p}] \right) (a_1 \cdots a_{p+q}) \left( \frac{\partial^q}{\partial y_1 \cdots \partial y_q} [a_{j_1} | \dots | a_{j_q}] \right), & \text{iff } a_i a_j = a_j a_i, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where  $\rho_{PQ} = \text{sign} \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & p+q \\ i_1 & \cdots & i_p & j_1 & \cdots & j_q \end{pmatrix}$ ,  $1 \leq i < j \leq p+q$ . This completes the proof.  $\square$

**Lemma 4.4.** *Let us suppose that all columns  $a_1, \dots, a_\ell$  are commutative, and let us suppose that for any  $1 \leq i, j \leq \ell$  we have  $\frac{\partial a_i}{\partial e_j} \neq 0$ , then  $\left( \frac{\partial}{\partial e_1} \smile \cdots \smile \frac{\partial}{\partial e_\ell} \right) [a_1 | \dots | a_\ell] = 0$ .*



*Proof.* First of all let us prove the following formulae,

$$\frac{\partial a}{\partial e_i} \frac{\partial b}{\partial e_j} = \begin{cases} \frac{\partial b}{\partial e_i} \frac{\partial a}{\partial e_j}, & \text{if } \{e_j\} \subseteq \{a\} \\ \frac{\partial b}{\partial e_j} \frac{\partial a}{\partial e_i}, & \text{if } \{e_j\} \subseteq \{b \setminus a\}, \end{cases} \quad i \neq j \quad (4.29)$$

Indeed, if  $\{e_j\} \subseteq \{a\}$  then

$$\frac{\partial a}{\partial e_i} \vee \frac{\partial b}{\partial e_j} = \{a \setminus e_i\} \cup \{b \setminus a\} \cup (\{a \setminus e_j\})^{\{a \setminus e_i\}} = \{a \setminus e_i\} \cup \{b \setminus a\} = \frac{\partial b}{\partial e_i},$$

and from (2.17) follows  $\frac{\partial a}{\partial e_i} \wedge \frac{\partial b}{\partial e_j} = \frac{\partial a}{\partial e_j}$ . Let  $\{e_j\} \not\subseteq \{a\}$ , then

$$\begin{aligned} \frac{\partial a}{\partial e_i} \vee \frac{\partial b}{\partial e_j} &= \{a \setminus e_i\} \cup ((\{b \setminus a\} \setminus \{e_j\}) \cup \{a\})^{\{a \setminus e_i\}} = (\{a \setminus e_i\}) \cup ((\{b \setminus a\} \setminus \{e_j\}) \cup \{e_i\}) = \\ &= \{a\} \cup ((\{b \setminus a\} \setminus \{e_j\})) = \{b \setminus e_j\} = \frac{\partial b}{\partial e_j}, \end{aligned}$$

and from (2.17) follows  $\frac{\partial a}{\partial e_i} \wedge \frac{\partial b}{\partial e_j} = \frac{\partial a}{\partial e_i}$ , as claimed.

Now, let us consider the sum

$$\left( \frac{\partial}{\partial e_{i_1}} \smile \dots \smile \frac{\partial}{\partial e_{i_\ell}} \right) [a_1 | \dots | a_\ell] = \sum_{\sigma \in \mathbb{S}_n} \text{sign}(\sigma) (a_\ell \dots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}},$$

from condition  $\frac{\partial a_i}{\partial e_j} \neq 0$  for any  $1 \leq i, j \leq \ell$  follows  $\{e_1\}, \dots, \{e_\ell\} \subseteq \{a_1\}$ , then using (4.29) we see that for any permutations  $\sigma, \pi$  we get  $\frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}} = \frac{\partial a_{\pi(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\pi(\ell)}}{\partial e_{i_\ell}}$ , i.e.,  $\left( \frac{\partial}{\partial e_1} \smile \dots \smile \frac{\partial}{\partial e_\ell} \right) [a_1 | \dots | a_\ell] = 0$ . q.e.d.  $\square$

**Theorem 4.2.** *For the plactic monoid algebra  $\mathbb{k}[\text{Pl}_n]$  the Hochschild cohomology algebra can be describe as below*

$$HH^*(\mathbb{k}[\text{Pl}_n]) \cong \bigwedge_{\mathbb{k}} [\alpha_1, \dots, \alpha_n] / (\alpha_i \alpha_j = 0, \text{ iff } a_i a_j \neq a_j a_i \text{ here } a_i, a_j \in \mathbb{I})$$

*Proof.* From Proposition 4.2 follows that it is enough to prove there are not another relations except skew commutativity. First of all let us remark from (4.1) follows the following property,

$$\text{if } \frac{\partial a_j}{\partial e_{i_k}} = 0, \text{ then } \frac{\partial a_{j-t}}{\partial e_{i_k}} = 0 \text{ for any } 1 \leq t \leq j. \quad (4.30)$$

Let us consider the sum

$$\left( \frac{\partial}{\partial e_{i_1}} \smile \dots \smile \frac{\partial}{\partial e_{i_\ell}} \right) [a_1 | \dots | a_\ell] = \sum_{\sigma \in \mathbb{S}_n} \text{sign}(\sigma) (a_\ell \dots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(k)}}{\partial e_{i_k}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}},$$

let us assume that  $\frac{\partial a_j}{\partial e_{i_j}} = 0$ . Consider now the set  $S_{jk} := \{\sigma \in \mathbb{S}_n : \sigma(k) = j - t, 1 \leq t \leq j - 1\}$ , then we have

$$\sum_{\sigma \in \mathbb{S}_n} \text{sign}(\sigma) (a_\ell \dots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(k)}}{\partial e_{i_k}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}} = \sum_{\sigma \in \mathbb{S}_n \setminus S_{jk}} \text{sign}(\sigma) (a_\ell \dots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(k)}}{\partial e_{i_k}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}},$$

We see that after repeating this procedure we'll get

$$\sum_{\sigma \in \mathbb{S}_n} \text{sign}(\sigma) (a_\ell \dots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(k)}}{\partial e_{i_k}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}} = \sum_{\sigma \in X} \text{sign}(\sigma) (a_\ell \dots a_1) \frac{\partial a_{\sigma(1)}}{\partial e_{i_1}} \dots \frac{\partial a_{\sigma(k)}}{\partial e_{i_k}} \dots \frac{\partial a_{\sigma(\ell)}}{\partial e_{i_\ell}},$$

from Lemma 4.4 follows this sum is non zero iff  $X$  consists only one permutation, q.e.d.  $\square$

**Remark 4.1.** *The rings of the form  $\bigwedge_R [\alpha_1, \dots, \alpha_m] / (\alpha_i \alpha_j = 0 \text{ iff } i, j \in J)$ , where  $J$  is a set, were considered in [25], where there were investigated diagrams associated with a finite simplicial complex in various algebraic and topological categories. These rings are called the Stanley — Reisner rings or rings of faces.*

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